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Brane world generation by matter and gravity

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Abstract: We present a non-compact *4 + 1* dimensional model with a local strong four-fermion interaction supplementing it with gravity. In the strong coupling regime it reveals the spontaneous translational symmetry breaking which eventually leads to the formation of domain walls, or thick 3-branes, embedded in the AdS_5 manifold. To describe this phenomenon we construct the appropriate low-energy effective Action and find kink-like vacuum solutions in the quasi-flat Riemannian metric. We discuss the generation of ultra-low-energy *3 + 1* dimensional physics and we establish the relation among the bulk five dimensional gravitational constant, the brane Newton's constants and the curvature of AdS_5 space-time. The plausible relation between the compositeness scale of the scalar matter and the symmetry breaking scale is shown to support the essential decoupling of branons, the scalar fluctuations of the brane, from the Standard Model matter, supporting their possible role in the dark matter saturation. The induced cosmological constant on the brane does vanish due to exact cancellation of matter and gravity contributions.

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1. Introduction

The conjecture about that our *3 + 1* dimensional world might be allocated on a brane (or domain wall) in a multi-dimensional space-time has recently invoked much activity [1]-[8], owing to the new tools it provides to solve the long standing mass and scale hierarchy problems [3]-[18] in particle theory. New extra dimensional physics could manifest itself in accessible experiments and observations, when the size of extra dimensions is relatively large [3, 5] or even infinite [1], [19] – [22], in as much to feed research programs in running and forthcoming collider and non-collider experiments [5], [23]–[28], [29], [30]. Nowadays the brane world scenarios and their applications are well summarized in a dozen of review articles [31]–[40].

On the one hand, the brane itself is often considered as an *elementary* geometrical object of a vanishing thickness along the extra dimensions, as it is promoted by superstring theories (with supersymmetric compactifications, see [32] and references therein, and non-supersymmetric ones, see [41] and references therein). In other words, this kind of brane represents just a boundary for higher dimensional objects. On the other hand, such an approach gives rise to the question about the origin of branes trapping our matter world.

In addition to, when endowing branes with a tiny thickness, one allows some more options for a solution of the mass hierarchy problem [17, 18, 42, 43] (see also reviews [31, 33, 39]).

A non-antagonistic alternative to the infinitely thin branes is provided by an effective multi-dimensional field theory, owing to the possibility of a spontaneous breaking of the translational symmetry. The thick (or fat) brane (or domain wall) formation and the trapping of light particles in its layer might be obtained [44]–[52] by a number of particular background scalar and/or gravitational fields living in the multi-dimensional bulk, when their vacuum configuration has a non-trivial topology, thereby generating zero-energy states localized on the brane.

Respectively, the mechanism of how such background fields might emerge and further induce the spontaneous breaking of translational symmetry is worthy to be elaborated and the domain wall creation, due to the self-interaction of certain particles in the bulk, may become a conceivable and appealing possibility [53].

In this paper we continue the exploration of a non-compact *4 + 1* dimensional fermion model [53] with a local strong four-fermion interaction, by supplementing it with a partially induced background gravitational field. Both kinds of interactions will lead coherently first to the discrete symmetry breaking and, further on, to the breaking of translational invariance. This can be achieved in terms of a particular domain wall pattern of the vacuum state [54]– [60], just allowing the light massive Dirac particles to live essentially in *3 + 1* dimensions. Those very same interactions generate localized zero-modes for the composite scalar fields, just completing the fermion matter content on the brane with a scalar counterpart.

We shall concentrate ourselves upon the main dynamical origin of the spontaneous symmetry breaking – on fermion self-interaction supplemented by gravity – and yet simplify the model by neglecting all gauge field interaction. In this sense, our model may be considered a sector of the so called *domain wall gravitating standard model*.

The five dimensional fermion model with spin-0 and spin-2 induced self-interaction is formulated in Section 2. There the fermion self-interaction *via* the spin-0 channel is restricted to a four-fermion type that will be sufficient to trigger the localization of massive Dirac fermions. It contains two dimensional coupling constants expressed in units of the compositeness scale Λ. The latter one plays the role of a cut-off for virtual fermion energies and momenta.

Meanwhile, the interaction *via* the spin-2 channel (*extra dimensional gravity*) is introduced in a non-linear way, in order to make this model covariant under the space-time diffeomorphisms, with a specific five dimensional Einstein-Hilbert bare Action and a bare cosmological constant to balance the formation of the physical Newton's and cosmological constants on the brane. Among different options, we pay attention to the case when the five dimensional gravity is fundamental and its related Einstein-Hilbert term is not too much corrected by fermion induced effective Action. As well, we shall turn ourselves to the economical scenario of induced gravity when, contrary to the previous case, the bare gravitational constant may be taken negligible and gravity is principally induced by fermion matter being therefore composite.

In this Section 2, once the scalar bosonization of the four-fermion interaction has been

implemented, the low-energy effective Action for composite scalar and gravity fields is obtained in the mean field or large-N approximation, where N roughly counts the number of fermion species in the Standard Model. This effective Action, which arises out of fermion one-loop radiative corrections, does accumulate the radiative contributions of high-energy fermion virtualities to describe infrared phenomena of spontaneous symmetry breaking. It already contains the kinetic terms for the scalar auxiliary fields, endowing them with the structure of composite fields. In the calculation of that effective Action, the Euclidean space-time approach for the invariant cut-off and the finite-mode regularization [61] for separation of high- and low-energy fermion fields are employed.

In Section 3 we search for classical vacuum configurations of gravity and scalar fields by analyzing the low-energy effective Action. This search is restricted to the class of conformal-like metrics (warped geometries) with the flat Minkowski hyperplanes at each point along the fifth coordinate. The joint solution of Einstein and non-linear Klein-Fock-Gordon equations is properly found within the weak-gravity approximation, *i.e.* assuming a relatively small five dimensional gravitational constant, what will be eventually justified in Section 4 after normalization to the Newton's gravitational constant on the brane. As expected to the leading order, the equations of motion for the scalar fields are not affected by gravity and their solutions coincide with the flat case investigated in [53]. In this sense, in the model under discussion, the brane and warped geometry creation is basically maintained by matter but not by gravity[1].

We select out the magnitudes of our four-fermion coupling constants in such a way that the dynamical breaking of the so called τ-symmetry and translational invariance were supported and, moreover, that the brane located Dirac fermions were supplied with masses. The characteristic scale M of the symmetry breaking – inverse thickness of a brane – can be conceivably much less than the compositeness scale Λ, keeping consistently the brane formation and particle localization phenomena in the low-energy region.

The remaining pair of the Einstein-like field equations can be rearranged so that one of them does specify the warp factor of the five dimensional anti-de Sitter (AdS_5) geometry, whereas the other one does represent the integral of motion with the bulk cosmological constant as an integration constant. The latter property holds exactly in all order of gravitational perturbation theory. As a consequence, this integration constant is found to be firmly fixed by the vacuum (kink-like) configurations of the scalar fields and of the warped geometry.

In Section 4 we examine the spectral properties of light particles allocated on the brane, we describe the structure of the ultra-low energy Lagrange density and finally summarize the results of [53] for the coupling constants of light particle interactions, which are parameterized by the ratio M/Λ of the symmetry breaking and the compositeness scales. The mass spectrum of light particle together with their interactions are described up to the leading order in the gravitational constant. We argue that this leading contribution into the brane world matter interaction is, in fact, independent of the five dimensional gravity,

[1]The interplay between gravity and matter effects in brane world formation and corresponding Einstein equations have been subjects of extensive studies in different models [1, 31, 35, 40, 44, 45, 46, 48, 62].

in spite of the nonperturbative quantum tunneling of massive states off the brane [51] in the AdS_5 geometry.

Section 5 is devoted to the estimation of all the scales and the coupling constants we have previously introduced. This can be achieved by imposing the observed value of the Newton's constant and the Newton's gravitational law within the presently available limits in modern experiments [63]. It is adopted that both the scale of compositeness and the translational symmetry breaking scale must be naturally very high, in between several TeV's and the GUT scale 10^{15} GeV. Therefrom it is derived that, if five dimensional gravity is fundamental and displaying some Planck-like scale much larger than the compositeness scale, then there is a window for a non-trivial (but weak) interaction between brane fluctuations (branons [29, 64, 65]), Higgs-like scalars and fermions, with a low compositeness scale of the order tens of TeV's, the trapping barrier for light particles as high as $2 \div 3$ TeV and the AdS_5 curvature scale $\sim 10^{-3}$ eV, which is comparable with the existing experimental checks of the Newton's law [63].

On the other hand, if the induced gravity is dominated at ultra-low energies, then the scalar matter interaction is highly suppressed, just making branons good sterile candidates to contribute into dark matter [29]. The cosmological constant induced on the brane is also calculated and found to vanish exactly, just making consistently endorsed the *ansatz* for the flat Minkowski's hyperplanes[2].

In the concluding Section we mainly summarize our observations on the interplay between the five dimensional Planck-like mass, the compositeness and the dynamical symmetry breaking scales and the AdS_5 curvature. As a further development, the possibility of implementing an occasional tiny matter and vacuum energy defect, located in the forth space direction in the presence of gravity, is shortly discussed. As a matter of fact, in our previous paper [53] we have demonstrated that such a defect may trigger and thereby justify the dynamical breaking of translational invariance at a particular place in the extra dimensions[3] and it might supply the branon degrees of freedom with a small mass.

2. Five dimensional fermion model with scalar and gravity induced self-interaction

Let us remind [53] the domain wall phenomenology and introduce the necessary notations. We start from the model of one four-component fermion bi-spinor field $\psi(X)$ defined on a five dimensional flat Minkowski space-time and coupled to a scalar field $\Phi(X)$. The extra-dimension coordinate is assumed to be space-like,

$$X^A = (x^\mu, z) \ , \quad x^\mu = (x^0, x^1, x^2, x^3) \ , \quad (\eta^{AA}) = (+,-,-,-,-)$$

and the subspace of coordinates x^μ eventually corresponds to the four dimensional Minkowski space. The extra-dimension size is supposed to be infinite (or large enough). The fermion

[2]This phenomenon of exact compensation between gravity and matter contributions into the brane cosmological constant has been also found in [1], [66] – [68] in different brane-world models.

[3]Another realization of thin defects in extra dimensional world see in [69].

wave function is then described by the Dirac equation

$$[\, i\gamma^A \partial_A - \Phi(X)\,]\psi(X) = 0\ , \quad \gamma^A = (\gamma^\mu, i\gamma_5)\ , \quad \{\gamma^A, \gamma^B\} = 2\eta^{AB}\ , \tag{2.1}$$

γ^μ being a standard set of four dimensional Dirac matrices in the chiral (or Weyl) representation with $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$.

The trapping of light fermions on a four dimensional hyper-plane – the domain wall – localized in the fifth dimension at $z = z_0$ can be promoted by a certain *topological*, z-dependent configuration of the vacuum expectation value of the scalar field $\langle\Phi(X)\rangle_0 = \varphi(z)$ – for instance $\varphi(z) = M\tanh(Mz)$ – due to the appearance of zero-modes with a certain chiralities in the spectrum of the four dimensional Dirac operator [1, 31].

If we aim to build up a light Dirac fermion, we need two different chiralities for the same shape of scalar background. Then the minimal set of fermions over the five dimensional space-time has to include [51, 53] two proto-fermions $\psi_1(X), \psi_2(X)$. In order to generate left- and right-handed parts of a four dimensional Dirac bi-spinor as zero modes, those fermions have to couple to the scalar field $\Phi(X)$ with opposite charges,

$$[\, i\not{\partial} - \tau_3\Phi(X)\,]\Psi(X) = 0\ , \quad \not{\partial} \equiv \widehat{\gamma}^A\partial_A\ , \quad \Psi(X) = \begin{pmatrix} \psi_1(X) \\ \psi_2(X) \end{pmatrix}\ , \tag{2.2}$$

where $\widehat{\gamma}^A \equiv \gamma^A \otimes \mathbf{1}_2$ are Dirac matrices and $\tau_a \equiv \mathbf{1}_4 \otimes \sigma_a,\ a = 1, 2, 3$ are the generalizations of the Pauli matrices σ_a acting on the bi-spinor components $\psi_i(X)$.

In addition to the trapping scalar field, a further one is required to supply light domain wall fermions with a mass. Its coupling must mix left and right chiralities as the mass term breaks the chiral invariance. Thus we introduce two types of four-fermion self-interactions to reveal two composite scalar fields with a proper coupling to fermions. These two scalar fields acquire mass spectra similar to fermions with light counterparts located on the domain wall. The dynamical scheme of creation of domain wall particles turns out to be quite economical and few predictions on masses and decay constants of fermion and boson particles have been derived [53]. However the allocation of matter on the domain wall certainly lead to strong gravitational effects. Moreover, the gravity itself may cause in turn the localization of the matter fields on a domain wall.

In the present paper we shall extend the dynamical mechanism of the fermion self-interaction by including, partially or completely, the gravitational contribution as induced by the high-energy spinor matter. Our main interest is focused on the thick brane formation corresponding to the flat *3+1* dimensional Minkowski space, whereas the gravity is non-trivial in the fifth direction orthogonal to the Minkowski's world. It is treated self-consistently together with the vacuum configurations of composite scalar fields.

Let us now formulate the fermion model in five dimensions[4], which implements the mechanism of translational symmetry breaking to create a domain wall. It is described by the classical Lagrange density

$$\mathcal{L}^{(5)}(\overline{\Psi}, \Psi) = \overline{\Psi}\, i\not{\partial}\Psi + \frac{g_1}{4N\Lambda^3}\left(\overline{\Psi}\tau_3\Psi\right)^2 + \frac{g_2}{4N\Lambda^3}\left(\overline{\Psi}\tau_1\Psi\right)^2, \tag{2.3}$$

[4]We extend the idea of the Top-mode Standard Model (TmSM) [70] on extra dimensions. Other extensions of TmSM have been undertaken in [71, 72].

where $\Psi(X)$ is an eight-component five dimensional fermion field, see eq. (2.2) – either a bi-spinor in a four dimensional theory or a spinor in a six-dimensional theory – which may also realize a flavor and color multiplet with the total number $N = N_f\, N_c$ of spinor degrees of freedom. We can say that, *grosso modo*, the number of color and flavor degrees of freedom of massive fermions in the Standard Model is around twenty, if all of them are originated from corresponding five dimensional proto-fermions. The ultraviolet cut-off scale Λ bounds fermion momenta, as the four-fermion interaction is supposed here to be an effective one, whereas g_1 and g_2 are suitable dimensionless and eventually scale dependent effective couplings.

This Lagrange density can be more conveniently represented with the help of a pair of auxiliary scalar fields $\Phi(X)$ and $H(X)$, which eventually will allow to trap a light fermion on the domain wall and to supply it with a mass: namely,

$$\mathcal{L}^{(5)}(\overline{\Psi}, \Psi, \Phi, H) = \overline{\Psi}(i\,\partial\!\!\!/ - \tau_3\Phi - \tau_1 H)\Psi - \frac{N\Lambda^3}{g_1}\,\Phi^2 - \frac{N\Lambda^3}{g_2}\,H^2\ . \tag{2.4}$$

For sufficiently strong couplings, this system undergoes the phase transition to the state in which the condensation of fermion-antifermion pairs does spontaneously break – partially or completely – the so-called τ-symmetry: $\Psi \longrightarrow \tau_1\Psi;\ \Phi \longrightarrow -\Phi$; and $\Psi \longrightarrow \tau_3\Psi;\ H \longrightarrow -H$.

In order to develop the infrared phenomenon of the τ-symmetry breaking, the effective Lagrange density containing essential low-energy degrees of freedom has to be suitably derived. To this concern, we proceed in the transition to the Euclidean space, where the invariant momentum cut-off can be unambiguously implemented. Within this framework, the notion of low-energy is referred to momenta $|p| < \Lambda_0$ as compared with the cut-off $\Lambda_0 \ll \Lambda$. However, after the elaboration of the domain wall vacuum, we will search for the fermion states with masses m_f much lighter than the dynamical scale Λ_0, *i.e.* for the ultralow-energy physics. Thus, eventually, there are three scales in the present model in order to implement the domain wall particle trapping.

Let us decompose the momentum space fermion fields into their high-energy part $\Psi_h(p) \equiv \Psi(p)\vartheta(|p| - \Lambda_0)\vartheta(\Lambda - |p|)$, their low-energy part $\Psi_l(p) \equiv \Psi(p)\vartheta(\Lambda_0 - |p|)$ and integrate out the high-energy part of the fermion spectrum, $\vartheta(t)$ being the usual Heaviside's step distribution.

The above decomposition of the fermion spectrum should be done covariantly, *i.e.* in terms of the full Euclidean Dirac's normal operator,

$$D\!\!\!\!/ \equiv i(\partial\!\!\!/ + \tau_3\Phi + \tau_1 H)\ . \tag{2.5}$$

As we want to investigate the τ-symmetry breaking by fermion condensation, in what follows we can neglect the high-energy components of the auxiliary boson fields, what is equivalent to adopt the mean-field or large N approximations. Then the low-energy Euclidean Lagrange density, which accounts for low-energy fermion and boson fields, can be written as the sum of the Euclidean counterpart of the classical Lagrange density (2.4) and the induced one-loop contribution, arising by functional integration of the high-energy

fermion field components: namely,

$$\mathcal{L}^{(5)}_{\rm low}(\overline{\Psi}_l,\Psi_l,\Phi,H)=\mathcal{L}^{(5)}_E(\overline{\Psi}_l,\Psi_l,\Phi,H)+\Delta\mathcal{L}^{(5)}(\Phi,H)\ , \tag{2.6}$$

where the classical Euclidean Lagrange density reads

$$\mathcal{L}^{(5)}_E(\overline{\Psi}_l,\Psi_l,\Phi,H)=\overline{\Psi}_l\,\not{D}\Psi_l+\frac{N\Lambda^3}{g_1}\,\Phi^2+\frac{N\Lambda^3}{g_2}\,H^2\ . \tag{2.7}$$

The one-loop contribution of high-energy fermions is given by

$$\begin{aligned}&\Delta\mathcal{L}^{(5)}(\Phi,H)=-(N/2)\,{\rm tr}\,\langle X|{\rm A}|X\rangle\ ,\\ &{\rm A}\equiv\vartheta(\Lambda^2-\not{D}^\dagger\not{D})\ln\frac{\not{D}^\dagger\not{D}}{\Lambda^2}-\vartheta(\Lambda_0^2-\not{D}^\dagger\not{D})\ln\frac{\not{D}^\dagger\not{D}}{\Lambda_0^2}\ ,\end{aligned} \tag{2.8}$$

where the symbol [tr] stands for the trace over spinor and internal degrees of freedom. In the latter operator [A] we have incorporated the cut-offs which select out the above defined high-energy region [61]. For $n=5$ we eventually found [53]

$$\begin{aligned}&\Delta\mathcal{L}^{(5)}(\Phi,H)\stackrel{\Lambda\to\infty}{\sim}\\ &\frac{N\Lambda}{4\pi^3}\left[\,(\partial_A\Phi)^2+(\partial_AH)^2\,\right]-\frac{N\Lambda^3}{9\pi^3}\left(\Phi^2+H^2\right)+\frac{N\Lambda}{4\pi^3}\left(\Phi^2+H^2\right)^2\ .\end{aligned} \tag{2.9}$$

Now, let us switch on gravity as described by the metric field $g_{AB}(X)$ on a five dimensional Riemannian manifold $\mathcal{M}_5$ which is called the base space. The capital Latin indexes $A,B,C,\ldots$ are usually called the base indexes or holonomic indexes. The Action is appropriately normalized with the help of the determinant of this metric,

$$S(\Phi,H,\overline{\Psi}_l,\Psi_l,g)=\int_{\mathcal{M}_5}d^5X\sqrt{g}\,\left[\mathcal{L}^{(5)}_{\rm fermion}+\mathcal{L}^{(5)}_{\rm boson}\right]\ ;\quad g\equiv\det(g_{AB})\ , \tag{2.10}$$

where the fermion (spinor) and boson (scalar and gravity) parts of the Lagrange density will be defined here below. Namely, for the Lagrange density (2.7) the form invariant under diffeomorphisms of the part bilinear in fermion fields is given in terms of the pentad–fields or *fünfbeine* $e^i_A(X)$, which connect locally the curved manifold $\mathcal{M}_5$ with the flat space with Euclidean signature, so that

$$e^i_A(X)e^i_B(X)=g_{AB}(X)\ ;\quad e^A_i(X)e^B_i(X)=g^{AB}(X)\ ;\quad e^i_A(X)e^A_j(X)=\delta^i_j\ , \tag{2.11}$$

where the Euclidean frame Latin small indexes, also called anholonomic, run from one to five, in such a way that

$$\{\widehat{\gamma}_j,\widehat{\gamma}_k\}=2\delta_{jk}\,\left(\mathbf{1}_4\otimes\mathbf{1}_2\right)\ . \tag{2.12}$$

The invariant spinor Lagrange density then reads

$$\begin{aligned}\mathcal{L}^{(5)}_{\rm fermion}(\overline{\Psi}_l,\Psi_l,\Phi,H,g)&=i\overline{\Psi}_l\left[\,\widehat{\gamma}_ke^A_k\left(\partial_A+\omega_A\right)+\tau_3\Phi+\tau_1H\,\right]\Psi_l\\ &=i\overline{\Psi}_l\left(\not{\nabla}+\tau_3\Phi+\tau_1H\right)\Psi_l\\ &\equiv\overline{\Psi}_l\,\not{\mathbf{D}}\Psi_l\ ,\end{aligned} \tag{2.13}$$

where the spin connection ω_A can be represented in terms of the *fünfbeine* and the affine connection

$$\Gamma^{C}_{AB} = \frac{1}{2} g^{CD} \left(\partial_A g_{BD} + \partial_B g_{AD} - \partial_D g_{AB} \right) \ . \tag{2.14}$$

The spin connection has the following form

$$\omega_A \equiv \frac{1}{8} \left[\widehat{\gamma}_i, \widehat{\gamma}_j \right] \Gamma^{ij}_{A} \ ; \quad \Gamma^{ij}_{A} = e^{Bj} \Gamma^{C}_{AB} e^{i}_{C} - e^{Bj} \partial_A e^{i}_{B} \ . \tag{2.15}$$

As before in eq. (2.7), the fermion self-interaction is induced by those Yukawa–like vertexes which are linear in the auxiliary scalar fields Φ, H. We remind that the symmetry breaking phase arises when the classical scalar interaction compensates large contributions $\sim \Lambda^3$ in the low-energy effective action (2.9) induced by high-energy fermions. When the bare dynamical gravity is added, one is expected to find similar large contributions $\sim \Lambda^5$ and $\sim \Lambda^3$ in the one-loop effective Action, which may be tuned to end up with a gravitational dynamics not severely suppressed by a very high cosmological constant and a very small Newton's constant. As we will see the zeroth- and first-order Seeley–Gilkey coefficients a_0, a_2 contain respectively the invariant measure $\sqrt{g}$ and the scalar curvature R , where the curvature scalar is defined in the conventional way [73, 74]. Actually, let us denote with

$$R^{A}{}_{BCD} = \partial_C \Gamma^{A}_{BD} - \partial_D \Gamma^{A}_{BC} + \Gamma^{E}_{BD} \Gamma^{A}_{EC} - \Gamma^{E}_{BC} \Gamma^{A}_{ED} \ ; \tag{2.16}$$

$$R_{BD} \equiv R^{A}{}_{BAD} \ ; \quad R \equiv g^{BD} R_{BD} \tag{2.17}$$

the Riemann curvature tensor, the Ricci tensor and the scalar curvature respectively.

Now, if we aim to compensate all the one-loop large contributions as induced by the fermionic matter, we must add the classical bosonic Euclidean Lagrange density

$$\mathcal{L}^{(5)}_{\rm boson}(\Phi, H, g) = N\Lambda^3 \left(\frac{\Phi^2}{g_1} + \frac{H^2}{g_2} \right) - \frac{\Lambda}{\mathcal{G}} \left(\varepsilon \frac{R}{2} - \lambda_0 \right) \ , \tag{2.18}$$

We notice that in eq. (2.18) the quantity $\lambda_0 \sim \Lambda^2$ stands for the bare cosmological constant of the five dimensional universe. In turn, the Newton–like constant $\mathcal{G} \sim 1/\Lambda^2$ does specify the strength of the five dimensional gravitational interaction and its order of magnitude will be suitably tuned later on.

Yet, with the help of the factor $\varepsilon = 0, \pm 1$ we reserve ourselves the possibility to regulate different physical options for bare gravity: namely, for $\varepsilon = \pm 1$ the bare Einstein-Hilbert Action is admittedly generated from a more fundamental theory. On the one hand, it turns out that either this bare Action screens the induced gravity contribution for $\varepsilon = -1$, just like in the spirit of phenomenological supersymmetry, or, conversely, it enhances the induced gravity effect, for $\varepsilon = 1$, so that the five dimensional gravity appears to be very weak. Both cases do imply some particular dynamics beyond the compositeness scale responsible for the formation of a pure gravitational interaction. On the other hand, the choice of $\varepsilon = 0$ does give rise to another scenario, in which gravity is entirely induced by fermionic matter, *i.e.* by proto-fermions in the present model. In this latter case, we see how the five dimensional gravitational fields will certainly result to be very weak as for $\varepsilon = 1$. We find this option quite interesting and we shall estimate the related scales of the

five dimensional bulk physics, providing eventually the usual Newton's gravity in our brane universe.

Now we want to calculate the low-energy effective Action in the curved five dimensional space owing to the presence of gravity. To this purpose, let us define the elliptic second-order operator having the same spectrum as the original Dirac operator in (2.13). We use the conjugation property $\not{\mathbf{D}}^\dagger = \tau_2 \not{\mathbf{D}} \tau_2$ which relates the diffeomorphism and frame covariant Dirac operator $\not{\mathbf{D}}$ to its hermitian conjugate with respect to the invariant scalar product, as defined by the invariant measure of eq.(2.10). Thus, the diffeomorphism and frame covariant Euclidean Dirac operator is still a normal operator, which has to be implemented in order to get a real effective Action and to define the spectral cut-offs with the help of the positive operator

$$\not{\mathbf{D}}^\dagger \not{\mathbf{D}} = (i \not{\nabla})^2 + \Phi^2(X) + H^2(X) - \tau_3 \not{\partial}\Phi(X) - \tau_1 \not{\partial}H(X) \ . \tag{2.19}$$

This elliptic second-order differential positive operator can be re-expressed in a quite general form [75],

$$\not{\mathbf{D}}^\dagger \not{\mathbf{D}} \ = \ - \, g^{AB}(X) D_A D_B + \mathcal{M}^2(X) \ \equiv \ - \, D^2 + \mathcal{M}^2(X) \tag{2.20}$$

where $\mathcal{M}^2(X)$ is a matrix–valued multiplicative (*i.e.* non–differential) operator, whereas the full covariant derivative for a mixed quantity f having anholonomic (left understood) and holonomic indexes is given by $D_A f^C = (\partial_A + \omega_A) f^C + \Gamma^C_{AB} f^B$. Notice that the full covariant Laplace operator D^2 reduces to the Laplace-Beltrami operator when acting on a scalar quantity

$$D^2 f(X) = g^{AB}(X)\partial_A\partial_B f(X) + \frac{1}{\sqrt{g(X)}}\,\partial_A \left[\, g^{AB}(X)\sqrt{g(X)}\,\right] \partial_B f(X) \ . \tag{2.21}$$

Explicit evaluation yields

$$\mathcal{M}^2(X) = \frac{R}{4} + \Phi^2(X) + H^2(X) - \tau_3 \not{\partial}\Phi(X) - \tau_1 \not{\partial}H(X) \ . \tag{2.22}$$

where the identity matrix $\widehat{\mathbf{1}} \equiv \mathbf{1}_4 \otimes \mathbf{1}_2$ is always left understood. Now, let us calculate the effective low energy Lagrange density induced by high energy proto–fermions. One can see that, in fact, the scale anomaly only contributes into $\Delta\mathcal{L}^{(5)}$, *i.e.* that part which depends upon the scales. Thus, equivalently,

$$\Delta\mathcal{L}^{(5)}(\Phi, H, g) = N \int_{\Lambda_0}^{\Lambda} \frac{dQ}{Q} \ \mathrm{tr}\langle X|\vartheta(Q^2 - \not{\mathbf{D}}^\dagger \not{\mathbf{D}})|X\rangle \ . \tag{2.23}$$

As we assume that the scalar fields carry momenta much smaller than the lower scale Λ_0, then the diagonal matrix element in the RHS of eq. (2.23) can be calculated with the help of the derivative expansion of the representation [61]

$$\mathrm{tr}\langle X|\vartheta(Q^2 - \not{\mathbf{D}}^\dagger \not{\mathbf{D}})|X\rangle = \int_{-\infty}^{+\infty} \frac{dt}{2\pi i}\ \frac{\exp\{it\}}{t - i\varepsilon}\ \mathrm{tr}\langle X| \ \exp\left\{-i \ \not{\mathbf{D}}^\dagger \not{\mathbf{D}}/Q^2\right\}|X\rangle \tag{2.24}$$

where X belongs to the base space, which is supposed to be a Riemannian manifold of dimension n. The trace in eq. (2.24) can be calculated by means of the heat kernel method (see its review [76]), as shown in Appendix A. For $n = 5$, only three heat kernel coefficients at most are proportional to non–negative powers of the large parameter Q,

$$\begin{aligned}
&\mathrm{tr}\langle X|\vartheta(Q^2 - \not{\mathcal{D}}^\dagger \not{\mathcal{D}})|X\rangle \approx \frac{Q^5}{15\pi^3} - \frac{Q^3}{3\pi^3}\left[\Phi^2(X) + H^2(X) + \frac{R(X)}{12}\right] \\
&+ \frac{Q}{4\pi^3}\left\{\partial_A\Phi(X)\partial^A\Phi(X) + \partial_A H(X)\partial^A H(X) + [\,\Phi^2(X) + H^2(X)\,]^2\right. \\
&+ \frac{1}{6}R(X)[\,\Phi^2(X) + H^2(X)\,] - \frac{1}{3}D^2[\,\Phi^2(X) + H^2(X)\,] - \frac{1}{60}D^2R(X) \\
&\left. - \frac{7}{720}R_{ABCD}(X)R^{ABCD}(X) - \frac{1}{90}R_{AB}(X)R^{AB}(X) + \frac{R^2(X)}{144}\right\} , \qquad (2.25)
\end{aligned}$$

where, for large scales $\Lambda_0 \ll Q < \Lambda$, the neglected terms rapidly vanish. Inserting the RHS of eq. (2.25) in eq. (2.23) and taking into account that $\Lambda_0 \ll \Lambda$, one can neglect the Λ_0–dependence and find, up to a total penta–divergence,

$$\begin{aligned}
\Delta\mathcal{L}^{(5)}(\Phi, H, g) \overset{\Lambda\to\infty}{\sim} \;& \frac{N\Lambda^5}{75\pi^3} - \frac{N\Lambda^3}{9\pi^3}\left(\Phi^2 + H^2 + \frac{R}{12}\right) \\
+\;& \frac{N\Lambda}{4\pi^3}\left\{\,\partial_A\Phi\,\partial^A\Phi + \partial_A H\partial^A H\right. \\
+\;& \left(\Phi^2 + H^2\right)^2 + \frac{R}{6}\left(\Phi^2 + H^2\right) + \frac{R^2}{144} \\
-\;& \left.\frac{1}{90}R_{AB}R^{AB} - \frac{7}{720}R_{ABCD}R^{ABCD}\right\} . \qquad (2.26)
\end{aligned}$$

Although the actual values of the coefficients might be regulator–dependent, as already noticed, the coefficients of the kinetic and quartic terms of the effective low energy Lagrange density are definitely equal, no matter how the latter is obtained from the basic Dirac operator of eq. (2.5).

3. Scalars and gravity: classical configurations

The interplay between different operators in the low-energy Lagrange density (2.6) may lead to different dynamical regimes, depending of τ-symmetry breaking. The low-energy Euclidean Lagrange density can be cast in the form

$$\begin{aligned}
\mathcal{L}^{(5)}_{\mathrm{low}}(\overline{\Psi}_l, \Psi_l, \Phi, H, g) &\equiv \mathcal{L}^{(5)}_{\mathrm{fermion}}(\overline{\Psi}_l, \Psi_l, \Phi, H, g) + \mathcal{L}^{(5)}_{\mathrm{boson}}(\Phi, H, g) + \Delta\mathcal{L}^{(5)}(\Phi, H, g) \\
&= i\overline{\Psi}_l(X)\,[\,\not\nabla + \tau_3\Phi(X) + \tau_1 H(X)\,]\,\Psi_l(X) \\
&+ \frac{N\Lambda}{4\pi^3}\left\{\partial_A\Phi(X)\partial^A\Phi(X) + \partial_A H(X)\partial^A H(X)\right. \\
&- \left. 2\Delta_1\Phi^2(X) - 2\Delta_2 H^2(X)\right\} - \frac{\Lambda}{2\kappa\mathcal{G}}\{\,R(X) - 2\lambda\,\} \\
&+ \frac{N\Lambda}{4\pi^3}\left[\,\Phi^2(X) + H^2(X)\,\right]\left\{\Phi^2(X) + H^2(X) + \frac{R(X)}{6}\right\}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{N\Lambda}{2880\pi^3}\left\{5R^2(X) - 8R_{AB}(X)R^{AB}(X)\right.\\
&- \left. 7R_{ABCD}(X)R^{ABCD}(X)\right\}
\end{aligned}
\tag{3.1}
$$

where the two mass scales Δ_i characterize the deviations from the critical point

$$
\Delta_i(g_i) = \frac{2\Lambda^2}{9g_i}\left(g_i - g_i^{\rm cr}\right)\,; \quad g_1^{\rm cr} = g_2^{\rm cr} = 9\pi^3\ . \tag{3.2}
$$

The *dressed* cosmological constant λ and the gravitational low-energy dimensionless parameter κ do arise as a net effect of the interplay between the classical and fermion induced contributions[5]. If we introduce some average compositeness scale

$$
\Lambda_c^3 \equiv N\Lambda^3/54\pi^3, \tag{3.3}
$$

together with a five dimensional counterpart of the Planck scale

$$
\hbar c\Lambda/\mathcal{G}\kappa \equiv M_*^3\ , \tag{3.4}
$$

then we can easily obtain

$$
\lambda = \kappa\lambda_0 + \frac{N\Lambda^4}{75\pi^3}\,\mathcal{G}\kappa\ , \tag{3.5}
$$

together with the relationship

$$
\kappa = \frac{1}{\varepsilon + N\Lambda^2\mathcal{G}/54\pi^3} \qquad \varepsilon\kappa = 1 - \left(\frac{\Lambda_c}{M_*}\right)^3 \tag{3.6}
$$

where κ must be positive for the gravitational interaction to be attractive. We shall be interested in different scenarios which one can qualitatively specify as follows:

(A) `Fundamental gravity`,
when the bare and dressed gravitational couplings are comparable

$$
\varepsilon = 1\ , \qquad \kappa \simeq 1\ , \qquad \text{so that} \qquad \Lambda_c \ll M_*\ . \tag{3.7}
$$

(B) `Induced gravity`,
when the bare gravitational Action is either absent or irrelevant whilst the fermion induced gravitational Action is dominant

$$
\varepsilon = 0\ , \qquad \text{and/or} \qquad \kappa \ll 1\ , \qquad \text{so that} \qquad \Lambda_c \simeq M_*\ . \tag{3.8}
$$

(C) `Strong gravity`,
when the bare gravitational Action is dominating with respect to the fermion induced gravitational Action

$$
\varepsilon = -1\ , \qquad \kappa > 1\ , \qquad \text{so that} \qquad \Lambda_c > M_*\ . \tag{3.9}
$$

[5] We remark that the combination of terms quadratic in the scalar and tensor curvatures does not form the Gauss-Bonnet gravity interaction [78] and, as well, the scalar curvature "mass" $R/6$ for scalar matter fields does not imply an additional conformal symmetry of Klein-Fock-Gordon equations for the five-dimensional manifold [79] .

Eventually, we shall demand for all scales to be much less than the five dimensional Planck-like mass, otherwise our effective model would go beyond its own essence and must be replaced by the still unknown genuine theory of quantum gravity. This criterion rather rules out the scenario (C) with $\kappa \gg 1, \quad \Lambda_c \gg M_*$ as the very concept of composite particles and fields far above the Planck scale certainly needs to engage quantum gravity. This is why we will not focus on it keeping the side of a weak quasi–classical gravity.

In what follows we assume that all the matter effective couplings substantially reduce the cut-off scale Λ to a low energy scale $M \ll \Lambda$ and, to be definite, we take $\Delta_1 = M^2$. Thus, in order to make the different parts of the equations of motion comparable in mass scales, we shall suitably tune in the sequel the different couplings of our five dimensional model. In particular, the bare cosmological constant must be negative $\lambda_0 < 0$ in order to provide a small constant $\lambda \ll \Lambda^2$.

Let us obtain brane-like solutions which describe the localization along the fifth coordinate z. In this paper we are interested in the static effects of particle trapping on a brane and this is why we shall consider here only the flat four dimensional space-time, with Euclidean signature to simplify our calculations. As a consequence, we restrict ourselves to analyze the quasi-flat Riemannian metric, the invariant line element of which can be suitably chosen as follows

$$ds^2 = g_{AB}(X)\, dX^A\, dX^B = \exp\{-2\rho(z)\}\, dx_\mu dx_\mu + dz^2 \tag{3.10}$$

with the Euclidean signature. The physical motivations for the above choice will be clear later on. The related invariant volume factor is $\sqrt{g} = \exp\{-4\rho(z)\}$. The scalar and gravity parts of the low-energy Lagrange density (3.1) for this metric reduce to (see the expressions for curvature terms in Appendix B)

$$\begin{aligned}
&\mathcal{L}^{(5)}_{\text{boson}}(\Phi, H, g) + \Delta\mathcal{L}^{(5)}(\Phi, H, g) \;= \\
&(N\Lambda/4\pi^3)\, \exp\{2\rho(z)\}\, [\partial_\mu\Phi(X)\partial_\mu\Phi(X) + \partial_\mu H(X)\partial_\mu H(X)] + (N\Lambda/4\pi^3)\;\times \\
&\times \Big\{ [\,\partial_z\Phi(X)\,]^2 + [\,\partial_z H(X)\,]^2 - 2\Delta_1\Phi^2(X) - 2\Delta_2 H^2(X) + \big[\,\Phi^2(X) + H^2(X)\,\big]^2 \Big\} \\
&+ (N\Lambda/3\pi^3)\; \big[\,\Phi^2(X) + H^2(X)\,\big]\, \{\rho''(z) - (5/2)\,[\,\rho'(z)\,]^2\,\} \\
&+ (\Lambda/\kappa\mathcal{G})\, \{-4\rho''(z) + \; 10\,[\,\rho'(z)\,]^2 \;+\; \lambda\} \\
&+ (N\Lambda/120\pi^3)\, \{2[\,\rho''(z)\,]^2 - 36\rho''(z)[\,\rho'(z)\,]^2 + 45[\,\rho'(z)\,]^4\}
\end{aligned} \tag{3.11}$$

where $\rho'(z) \equiv d\rho/dz\,,\; \rho''(z) \equiv d^2\rho/dz^2$. Let us search for classical solutions in the form of a metric (3.10) in the gravitational weak coupling regime in which we assume that all along the large extra dimension we have

$$|\rho'(z)|/M = \mathrm{O}(\kappa)\;, \qquad |\rho''(\mathrm{z})|/\mathrm{M}^2 = \mathrm{O}(\kappa)\;. \tag{3.12}$$

This means that the conformal function $\rho(z)$ is slowly varying over a distance, in the extra dimension, of the order of magnitude of the typical Compton wavelength of the low energy particles. This requirement, together with $0 < \kappa \ll 1$ and $M \ll \Lambda$, yields that the last line in eq. (3.11), which corresponds to the terms quadratic in the curvature scalar

and curvature tensors of the Lagrange density (3.1), will be sub–leading and negligible with respect to the other terms. Nevertheless, just for completeness, we shall display the contribution of those terms into the equations of motion in Appendix C.

Within the present gravitational weak coupling approximation, the leading dynamics is mainly determined by the Einstein–Hilbert Action, on the five dimensional Riemannian manifold $\mathcal{M}_5$, and its coupling to the scalar matter fields. As shown in Appendix D, the field equations become

$$R_{AB} - \frac{1}{2}\, g_{AB}\, (R - 2\lambda) \;\equiv G_{AB} \;+\; \lambda\, g_{AB} \;=\; \frac{N\kappa\mathcal{G}}{2\pi^3}\, t_{AB} \tag{3.13}$$

where the normalized energy–momentum tensor of the scalar matter reads

$$\begin{aligned} t_{AB} \;&\equiv\; (4\pi^3/N\Lambda)\, T_{AB} \;\equiv\; \partial_A\Phi\, \partial_B\Phi + \partial_A H\, \partial_B H \\ &-\; \frac{1}{2}\, g_{AB} \left[\partial_C\Phi\, \partial^C\Phi + \partial_C H\, \partial^C H - 2\Delta_1\Phi^2 - 2\Delta_2 H^2 + \left(\Phi^2 + H^2\right)^2 \right] \\ &+\; \frac{1}{6} \left(R_{AB} - \frac{1}{2}\, g_{AB}\, R + g_{AB}\, D^C \partial_C - D_B\, \partial_A \right) \left(\Phi^2 + H^2\right) \;. \end{aligned} \tag{3.14}$$

The equations of motion for the scalar fields read

$$\begin{aligned} 2 \left[\Delta_1 - \Phi^2 - H^2 \right] \Phi \;&=\; \left(\frac{R}{6} \;-\; \frac{1}{\sqrt{g}}\, \partial_C \sqrt{g}\; g^{CD} \partial_D \right) \Phi \;, \\ 2 \left[\, \Delta_2 - H^2 - \Phi^2 \,\right] H \;&=\; \left(\frac{R}{6} \;-\; \frac{1}{\sqrt{g}}\, \partial_C \sqrt{g}\; g^{CD} \partial_D \right) H \;. \end{aligned} \tag{3.15}$$

In the case of a quasi–flat metric (3.10) and for kink–like profiles of the vacuum expectation values of the scalar fields $\langle \Phi(X) \rangle_0 = \Phi(z)$, $\langle H(X) \rangle_0 = H(z)$ one finds the following equations: namely,

$$G_{\alpha\alpha}(z) \;+\; g_{\alpha\alpha}(z)\, \lambda \;=\; \left(N\kappa\mathcal{G}/2\pi^3 \right)\, t_{\alpha\alpha}(z) \;, \tag{3.16}$$

$$G_{55}(z) \;+\; \lambda \;=\; \left(N\kappa\mathcal{G}/2\pi^3 \right)\, t_{55}(z) \;; \tag{3.17}$$

which respectively reduce to

$$\begin{aligned} \rho'' - 2\rho'^2 - \frac{1}{3}\, \lambda = \frac{N\kappa\mathcal{G}}{12\pi^3} \Big\{ & \Phi'^2 + H'^2 - 2\Delta_1\Phi^2 - 2\Delta_2 H^2 + \left(\Phi^2 + H^2\right)^2 \\ &+\; \left(\rho'' - 2\rho'^2 - \frac{1}{3}\, \frac{d^2}{dz^2} + \rho'\, \frac{d}{dz} \right) \left(\Phi^2 + H^2\right) \Big\} \end{aligned} \tag{3.18}$$

$$\begin{aligned} 2\rho'^2 + \frac{1}{3}\, \lambda = \frac{N\kappa\mathcal{G}}{12\pi^3} \Big\{ & \Phi'^2 + H'^2 + 2\Delta_1\Phi^2 + 2\Delta_2 H^2 \\ &-\; \left(\Phi^2 + H^2\right)^2 + \left(2\rho'^2 - \frac{4}{3}\, \rho'\, \frac{d}{dz} \right) \left(\Phi^2 + H^2\right) \Big\} \;. \end{aligned} \tag{3.19}$$

The field equations for the scalar matter become

$$\Phi'' = 2\Phi \left(\Phi^2 + H^2\right) - 2\Delta_1\Phi + 4\rho'\, \Phi' + \frac{2}{3}\, \Phi \left(2\rho'' - 5\rho'^2\right) \;, \tag{3.20}$$

$$H'' = 2H\left(\Phi^2 + H^2\right) - 2\Delta_2 H + 4\rho' H' + \frac{2}{3} H\left(2\rho'' - 5\rho'^2\right) \ . \tag{3.21}$$

Now, it turns out to be convenient to put forward the role of the low–energy relevant mass scale $\Delta_1 = M^2$. This can be better achieved after a suitable redefinition of the gravitational low–energy dimensionless parameter

$$\overline{\kappa} \ \equiv \ \frac{N\kappa}{6\pi^3} M^2 \mathcal{G} \ \ll \ 1 \tag{3.22}$$

and of the cosmological constant

$$\lambda \ \equiv \ 3\overline{\kappa}\, \lambda_{\text{eff}} \ . \tag{3.23}$$

It is worthwhile to notice that the sum of Eqs. (3.20) and (3.21) does not include the five dimensional rescaled cosmological constant λ_{eff} and reads

$$\rho'' = \frac{\overline{\kappa}}{M^2}\left\{\Phi'^2 + H'^2 + \frac{1}{2}\left(\rho'' - \frac{1}{3}\frac{d^2}{dz^2} - \frac{1}{3}\rho'\frac{d}{dz}\right)\left(\Phi^2 + H^2\right)\right\} \ . \tag{3.24}$$

On the other hand, eq. (3.19) can be rewritten in the form

$$\begin{aligned} 2M^2\lambda_{\text{eff}} \ &= \Phi'^2 + H'^2 + 2\Delta_1\Phi^2 + 2\Delta_2 H^2 - \left(\Phi^2 + H^2\right)^2 \\ &+ \left(2\rho'^2 - \frac{4}{3}\rho'\frac{d}{dz}\right)\left(\Phi^2 + H^2\right) - \frac{4M^2}{\overline{\kappa}}\rho'^2 \end{aligned} \tag{3.25}$$

which actually represents the integral of motion with the rescaled five dimensional cosmological constant playing the role of an integration constant. This can be checked by differentiating the above equation and taking into account Eqs. (3.20), (3.21) and (3.24).

We can approximate our field equations keeping in mind that we still assume

$$\frac{|\rho'(z)|}{M} \ = \ \mathrm{O}(\overline{\kappa}) \ = \ \frac{|\rho''(\mathrm{z})|}{\mathrm{M}^2} \ , \tag{3.26}$$

as previously specified. We emphasize that the validity of perturbation theory in $\overline{\kappa}$ is endorsed by the choice of a reference frame where the metric takes the form (3.10). For different choices such as, for instance, the conformal metric

$$ds^2 = \exp\{-2\sigma(y)\}\left(dx_\mu dx_\mu + dy^2\right) \ ; \quad \exp\{-\sigma(y)\}dy \equiv dz \ , \tag{3.27}$$

the expansion in $\overline{\kappa}$ is not uniform and fails for large z.

On the other hand, the expansion in powers of $\overline{\kappa}$ of the solutions of Eqs. (3.20), (3.21), (3.24) and (3.25) is well defined on the whole extra dimension $-\infty < z < \infty$. Let us find solutions by expanding in $\overline{\kappa} \ll 1$. According to our approximation (3.26), taking into account that $\Phi/M = \mathrm{O}(1) = H/M$, we obtain in the leading order

$$\frac{\rho''}{M^2} = \frac{\overline{\kappa}}{M^4}\left\{\Phi'^2 + H'^2 - \frac{1}{6}\frac{d^2}{dz^2}\left(\Phi^2 + H^2\right)\right\} + \mathrm{O}(\overline{\kappa}^2) \ , \tag{3.28}$$

whereas the cosmological constant to the lowest order reads

$$\frac{\lambda_{\text{eff}}}{M^2} \ = \frac{1}{2M^4}\left\{\Phi'^2 + H'^2 + 2\Delta_1\Phi^2 + 2\Delta_2 H^2 - \left(\Phi^2 + H^2\right)^2\right\} + \mathrm{O}(\overline{\kappa}) \ . \tag{3.29}$$

The latter equation firmly determines the five dimensional cosmological constant $\lambda_{\rm eff}$ in terms of the parameters for the kink–like solutions in the flat space which read

$$\begin{aligned} &\Phi'' + 2\,\Phi\left(\Delta_1 - \Phi^2 - H^2\right) = \mathrm{O}(\overline{\kappa})\ , \\ &H'' + 2H\left(\Delta_2 - \Phi^2 - H^2\right) = \mathrm{O}(\overline{\kappa})\ . \end{aligned} \tag{3.30}$$

To sum up, we have four equations for three functions $\rho(z), \Phi(z), H(z)$ and one integration constant $\lambda_{\rm eff}$. As in [53] one can discover two types of kink–like solutions for Eqs. (3.30): namely,

$$(J) \quad \Phi_J \equiv \langle\Phi(X)\rangle_0 = M\tanh(Mz)\ , \quad H_J \equiv \langle H(X)\rangle_0 = 0\ ; \tag{3.31}$$

$$(K) \quad \Phi_K \equiv \langle\Phi(X)\rangle_0 = M\tanh(\beta z)\ , \quad H_K \equiv \langle H(X)\rangle_0 = \mu\,\mathrm{sech}(\beta z)\ , \tag{3.32}$$

where

$$\mu = \sqrt{2\Delta_2 - M^2}\ , \qquad \beta = \sqrt{M^2 - \mu^2}\ . \tag{3.33}$$

The solution (K) exists only for $\Delta_2 < M^2 < 2\Delta_2$ and it coincides with the extremum (J) in the limit $\Delta_2 \to M^2/2,\ \mu \to 0,\ \beta \to M$. Both of them give consistently

$$\lambda = \frac{N\mathcal{G}\kappa M^4}{4\pi^3} = \frac{3}{2}\overline{\kappa}M^2\ . \tag{3.34}$$

The last term in (3.28) is originated from the scalar curvature mass–like term $R/6$ in the Lagrange density (3.1) for the scalar fields. To unravel its role, we split the conformal factor in the following way: namely,

$$\rho(z) = \rho_1(z) + \rho_2(z) = \frac{2\overline{\kappa}}{3}\left(1 + \frac{\mu^2}{2M^2}\right)\ln\cosh(\beta z) + Bz\ ; \tag{3.35}$$

$$\begin{aligned} \rho_1''(z) &= \frac{\overline{\kappa}}{M^2}\left\{[\,\Phi'(z)]^2 + [\,H'(z)]^2\right\}\ ; \quad \rho_2''(z) = \frac{-\overline{\kappa}}{6M^2}\,\frac{d^2}{dz^2}\left[\,\Phi^2(z) + H^2(z)\,\right]\ ; \\ \rho_1(z) &= \frac{2\overline{\kappa}}{3}\left(1 + \frac{\mu^2}{2M^2}\right)\ln\cosh(\beta z) + \frac{\overline{\kappa}}{6}\left(1 - \frac{\mu^2}{M^2}\right)\tanh^2(\beta z) + Bz\ ; \\ \rho_2(z) &= \frac{-\overline{\kappa}}{6M^2}\left[\,\Phi^2(z) + H^2(z) - \mu^2\,\right] \ = \ -\ \frac{\overline{\kappa}}{6}\left(1 - \frac{\mu^2}{M^2}\right)\tanh^2(\beta z)\ . \end{aligned} \tag{3.36}$$

This solution is normalized so that for the vanishing integration constant $B \to 0$ the function $\rho(z)$ becomes even and $\rho(0) = 0$. The latter corresponds to the proper normalization of the *3+1* metric on the brane $z = 0$.

One can see that the scalar curvature term substantially simplifies the metric factor $\rho(z)$ around the brane. Evidently, this solution approaches, when $B = 0$, the symmetric Anti-de-Sitter (AdS) metric for large z: namely,

$$\rho(z) \overset{|z|\to\infty}{\sim} k|z|\ ; \quad k \equiv \frac{2}{3}\,\overline{\kappa}\beta\left(1 + \frac{\mu^2}{2M^2}\right)\ \approx\ \frac{2}{3}\,\overline{\kappa}M\ . \tag{3.37}$$

4. Ultra-low energy physics in the matter sector

Let us summarize the structure [53] of the spectrum and of the interaction of the light states trapped on a brane, in the absence of gravity. The kinetic operators (second variation of the Action) of the two scalars $\Phi(X)$ and $H(X)$ and of the spinor field $\Psi(X)$ do exhibit normalizable zero-modes in the extra dimension, in the vicinity of the vacuum background (3.31) or (3.32), at the scaling point $M^2 = \Delta_1 = 2\Delta_2$ or $\mu = 0$. Those zero-modes $\phi_0(z)$, $h_0(z)$ and $\psi_0(z)$, respectively, are localized at the origin of the z-axis, with a localization width $\sim 1/M$ and, at ultra-low energies, the fluctuations of the matter fields can be parametrized as follows: namely,

$$
\begin{aligned}
\Phi(X) &\simeq \langle \Phi(X) \rangle_0 + \phi(x)\phi_0(z) \ ; \\
H(X) &\simeq \langle H(X) \rangle_0 + h(x)h_0(z) \ ; \\
\Psi(X) &\simeq \psi(x)\,\psi_0(z) \ .
\end{aligned}
\tag{4.1}
$$

For these states the ultra-low-energy effective Lagrange density (still in the Euclidean space) is generated

$$
\begin{aligned}
\mathcal{L}^{(4)} &= i\overline{\psi}(x)\,[\,\partial\!\!\!/ + g_f\,h(x)\,]\,\psi(x) + \frac{1}{2}[\,\partial_\mu\phi(x)\,]^2 + \frac{1}{2}[\,\partial_\mu h(x)\,]^2 \\
&+ \lambda_1\phi^4(x) + \lambda_2\phi^2(x)h^2(x) + \lambda_3 h^4(x) \ ,
\end{aligned}
\tag{4.2}
$$

with the ultra-low energy effective couplings given by

$$
g_f = \frac{\pi}{4}\sqrt{\zeta} \ , \quad \lambda_1 = \frac{18}{35}\zeta \ , \quad \lambda_2 = \frac{2}{5}\zeta \ , \quad \lambda_3 = \frac{1}{3}\zeta \ , \qquad \zeta \equiv \frac{M\pi^3}{\Lambda N} \ . \tag{4.3}
$$

Once gravity is switched on, it can be shown[6] that the zero-modes remain localizable – see below – and therefore the AdS vacuum solution does not play any dominant role concerning the determination of the coupling constants in eq. (4.2).

The situation becomes more subtle when light massive states are there. In the absence of gravity, the deviations off the scaling point towards the (K) vacuum configuration with $\mu \ll M$ do produce the masses for the Higgs-like particle and the spinor particles. Furthermore, the scalar self-interaction is induced in the form

$$
\begin{aligned}
\Delta\mathcal{L}^{(4)}_\mu &= \frac{1}{2}\,m_h^2\,h^2(x) + im_f\,\overline{\psi}(x)\psi(x) + \lambda_4\,h^3(x) \ ; \\
m_h^2 &= \mu\sqrt{2} \ ; \qquad m_f = \frac{\pi}{4}\,\mu \ ; \qquad \lambda_4 = \mu\sqrt{\zeta} \ .
\end{aligned}
\tag{4.4}
$$

On the one hand, we see that all interaction vertexes are governed by the parameter $\zeta \sim M/\Lambda$ and if $\zeta \ll 1$ the scalar matter essentially decouples from the fermion sector and does not interact without gravity. However, the parameter ζ is not properly fixed if gravity is not present and, in general, it is constrained by experimental bounds. On the other hand, the masses of Higgs-like scalar and fermions are controlled by the ultra-low scale μ

[6] It has been firstly analyzed in [27, 28, 51]. A detailed analysis more suitable for our model will be presented elsewhere.

independently of ζ. In the spirit of the Top-mode Standard Model [70] one expects that the heaviest quark mostly contributes into dynamical symmetry breaking, thereby driving to the scale $\mu \sim m_{\rm top} \sim 200$ GeV, *i.e.* the order of magnitude of the electroweak symmetry breaking scale [77].

Once the five dimensional gravity is switched on and the background geometry becomes an anti-de Sitter one, it turns out that light massive scalar and fermion 1-particle states belong, in fact, to the continuous part of the Hamiltonian spectrum, the latter ones being strongly localized on the brane albeit possessing a tiny non-vanishing tail for $|z| \gg 1/k$. The origin of this phenomenon can be schematically sketched in terms of the mass operator for a one dimensional scalar field living on the fifth dimension (see eq. (3.11) and [53]): namely,

$$\begin{aligned} \exp\{-2\rho(z)\}\mathbf{M}_z &\equiv -\,\partial_z[\,\exp\{-4\rho(z)\}\partial_z\,] + \mathbf{V}(z)\exp\{-4\rho(z)\}\ ; \\ \mathbf{M}_z\,\varphi(z) &= m^2\varphi(z)\ , \end{aligned} \tag{4.5}$$

where $\mathbf{V}(z)$ is, in general, some matrix-valued potential well giving rise to bound states, in the absence of gravity, and approaching the energy $4M^2$ for large z. The mass eigenvalue m stands for the mass of a scalar Higgs-like 1-particle state. After the suitable redefinition $\varphi(z) = \widetilde{\varphi}(z)\ \exp\{2\rho(z)\}$, the eigenvalue problem is transformed into a zero-mode condition $\widetilde{\mathbf{M}}_z\,\widetilde{\varphi}(z) = 0$, where the modified mass operator takes the conventional form,

$$\widetilde{\mathbf{M}}_z = -\,\partial_z^2 + \widetilde{\mathbf{V}}(z) - m^2\exp\{2\rho(z)\}\ \equiv\ -\,\partial_z^2 + \mathbf{W}(z)\ , \tag{4.6}$$

whereas $\widetilde{\mathbf{V}}(z) \simeq \mathbf{V}(z)$ uniformly up to the leading order in $\overline{\kappa}$. Thus we see how the only important difference in the spectral problem, in comparison with the five dimensional flat space, does consist in the appearance of the conformal factor, which is boundless increasingly when $|z| \gg 1/k$. The very last term in eq. (4.6) becomes now a piece of the modified potential $\mathbf{W}$, whilst the mass eigenvalue m just appears as a coupling constant.

Evidently, this very last piece, being negative, makes the modified potential unbounded from below: instead of a potential well, some finite barrier just arises. As a consequence, each light massive state with $0 < m^2 \ll 4M^2$ is at most quasi-stationary. Nonetheless, let us assign massive scalar particles in the brane world to be described by normalizable wave packets with a width of the order $1/M$. The latter ones are not, of course, exact eigenfunctions of the mass operator (4.6) but may be taken as close as possible, *e.g.* with the help of a variational principle, to the solutions in the vicinity of the brane $|z| \ll 1/k$. For instance, one can start with bound state wave functions in the absence of gravity. Then quantum mechanics predicts the decay rates of such a kind of bound states, due to quantum tunneling. The quasi-classical probability of barrier penetration is governed by the change in the sub-barrier wave function between the two turning points for classical trajectories, say, $0 < z_0 < z_1$. These turning points can be estimated to be

$$z_0 = \frac{C}{M}\ , \quad C \simeq 1\ ; \qquad z_1 \simeq \frac{1}{k}\ln\frac{2M}{m}\ , \tag{4.7}$$

so that $z_1 \longrightarrow \infty$ for the zero-modes $m \longrightarrow 0$. Taking these relations into account, one finds the suppression factor for quantum tunneling

$$\exp\left\{-\int_{z_0}^{z_1} dz' \sqrt{\mathbf{W}(z')}\right\} \simeq \exp\{-z_1 \cdot 2M\} \simeq \exp\left\{-\frac{3}{\overline{\kappa}} \ln \frac{2M}{m}\right\} , \tag{4.8}$$

where the limit $\mathbf{W} \longrightarrow 4M^2$ for $z_0 \ll z' \ll z_1$ and the definition (3.37) have been used.

As we shall discuss in the next Section, some reasonable order of magnitude for the weak gravitational constant is $\overline{\kappa} \leq 10^{-8}$, so that it is clear that such a suppression factor (4.8) is extremely small and spoils in practice any chance for matter to disappear from the brane. We also notice that the suppression factor is essentially non-perturbative and cannot be recovered as an expansion in powers of $\overline{\kappa}$ – see *e.g.* [80] for a complete discussion of the similar case of the Stark effect.

Now we can outline our further strategy in calculating gravitational effects on the mass spectrum and coupling constants of light particles living on the brane. First, particle wave functions at the lowest order are taken from the flat space limit. Second, only perturbation theory in powers of $\overline{\kappa}$ is involved in corrections of particle characteristics due to non-trivial gravitational background. In this way, our approach is able to implement the systematic perturbative expansion which provides light particle wave packets localized upon the brane, to any finite order in the expansion in powers of $\overline{\kappa}$. A more detailed and rigorous treatment is postponed to the forthcoming paper.

5. Newton's constant and parameters

One can find the relation between the five dimensional and brane gravity constants using the factorized Riemannian metric

$$ds^2 = \exp\{-2\rho(z)\}\, g_{\mu\nu}(x) dx^\mu dx^\nu + dz^2 . \tag{5.1}$$

For this metric, according to eq. (B.5) of Appendix B, the effective four dimensional Einstein-Hilbert Action, at the leading order, becomes

$$\begin{aligned} S[\,g\,] &= -\frac{\Lambda}{2\kappa\mathcal{G}} \int d^5X \sqrt{g(X)}\, \{R(X) \ - \ 2\lambda\} \\ &\simeq -\frac{\Lambda}{2\kappa\mathcal{G}} \int d^4x \sqrt{g(x)}\, R(x) \int_{-\infty}^{+\infty} dz\ \exp\{-2\rho(z)\} \\ &\quad -\frac{\Lambda}{\kappa\mathcal{G}} \int d^4x \sqrt{g(x)} \int_{-\infty}^{+\infty} dz\ \exp\{-4\rho(z)\} \left\{6[\,\rho'(z)\,]^2 - \lambda\right\} \\ &\equiv -\frac{1}{16\pi G_N} \int d^4x \sqrt{g(x)}\, \{R(x) \ - \ 2\Lambda_{\rm grav}\} , \end{aligned} \tag{5.2}$$

whence we eventually get the Planck mass scale $M_P \sim 1.22 \times 10^{19}$ GeV/c^2 which corresponds to the Newton's gravitational constant

$$M_P^2 = G_N^{-1} \equiv \frac{8\pi\Lambda}{\kappa\mathcal{G}} \int_{-\infty}^{+\infty} dz\ \exp\{-2\rho(z)\} \tag{5.3}$$

and the gravitational part of the four dimensional cosmological constant

$$\Lambda_{\rm grav} \equiv \frac{8\pi G_N \Lambda}{\kappa \mathcal{G}} \int_{-\infty}^{+\infty} dz \;\; \exp\{-4\rho(z)\} \left\{\lambda - 6[\,\rho'(z)\,]^2\right\} \;\; . \tag{5.4}$$

Under the assumption of the smallness of the parameter $\overline{\kappa} \ll 1$ and using in eq. (5.2) the approximate form for $|z| \to \infty$ of the solution (3.35) one obtains

$$G_N \;\simeq\; \frac{\pi^2\,\overline{\kappa}^{\,2}}{2N\Lambda M} \;\; . \tag{5.5}$$

Remarkably, the full value of the cosmological constant, including the gravitational as well as the matter vacuum energy densities, does indeed exactly vanish to all orders in the perturbative expansion in powers of $\overline{\kappa}$: actually,

$$\begin{aligned} \Lambda_{\rm cosmo} \equiv\; & \frac{2N\Lambda G_N}{\pi^2} \int_{-\infty}^{+\infty} dz \;\; \exp\{-4\rho(z)\} \Big\{ 2M^2\lambda_{\rm eff} - (4/\overline{\kappa})\, M^2 \rho'^{\,2}(z) \\ & + \Phi'^{\,2}(z) + H'^{\,2}(z) - 2\Delta_1 \Phi^2(z) - 2\Delta_2 H^2(z) + \left[\, \Phi^2(z) + H^2(z) \,\right]^2 \\ & + \frac{2}{3} \left[\, \Phi^2(z) + H^2(z) \,\right] \left[\, 2\rho''(z) - 5\rho'^{\,2}(z) \,\right] \Big\} \\ =\; & \frac{4N\Lambda G_N}{\pi^2 \overline{\kappa}} \int_{-\infty}^{+\infty} dz \;\; \exp\{-4\rho(z)\} \Big\{ -\; M^2 \rho''(z) + \overline{\kappa} \left[\, \Phi'^{\,2}(z) + H'^{\,2}(z) \,\right] \\ & + \frac{2}{3}\, \overline{\kappa} \left[\, \Phi^2(z) + H^2(z) \,\right] \left[\, 2\rho''(z) - 5\rho'^{\,2}(z) \,\right] \Big\} \;\; = \;\; 0 \;, \end{aligned} \tag{5.6}$$

where the vacuum expectation values (3.20) and (3.21) of the scalar fields together with the field equation (3.24) of the conformal factor have been suitably taken into account. In order to establish the exact cancellation between the gravitational and the scalar matter contributions, the five dimensional cosmological constant has been conveniently substituted from eq. (3.25). We also notice that the inclusion of the higher-order gravitational interaction in the last line of eq. (3.11) still keeps equal to zero the value of the induced cosmological constant $\Lambda_{\rm cosmo}$, as it can be shown with the help of the Appendix C.

Let us find the relations among the AdS curvature scale $k \simeq 2\overline{\kappa}\, M/3$, the Planck mass M_P and the spontaneous symmetry breaking scale M . We recall that the characteristic parameter of ultra-low energy dynamics of light fermions and scalar fields has been found [53] to be

$$\zeta \equiv \frac{M\pi^3}{N\Lambda} \;\; , \tag{5.7}$$

with $N \sim \pi^3$ in the Standard Model (see the summary in the previous Section 4). Therefore, the relationship among the three scales M_P, k and M actually reads

$$k^2 M_P^2 = \frac{8\pi}{9\zeta}\, M^4 \;\; , \tag{5.8}$$

in accordance with Eqs. (3.22) and (3.37). Correspondingly one obtains that

$$\bar{\kappa} = \frac{2\pi}{\sqrt{\zeta}}\, (M/M_P) \tag{5.9}$$

restricted to $\bar{\kappa} \ll 1$.

One can now express both the five dimensional gravitational constant and the five dimensional Planck scale, respectively, just in terms of the above mentioned scales and parameters, that means

$$\frac{\mathcal{G}\kappa}{\Lambda} = \frac{6\pi^3\overline{\kappa}}{N\Lambda M^2} = \frac{9k\zeta}{M^4} = \frac{1}{M_*^3}\ ; \quad M_*^3 = \frac{kM_P^2}{8\pi}\ , \tag{5.10}$$

which leads to the lower bound for the five dimensional Planck scale $M_* > 10^8$ GeV from the experimental bound $k > 10\ \mathrm{mm}^{-1} = 2\cdot 10^{-3}$ eV, the latter one being based on possible deviations from the Newton's law [63].

Our first scenario – `fundamental gravity` – is selected to have a principally detectable dynamics of scalar fields and of the Higgs-Yukawa coupling to fermions in the brane world. Let us therefore analyze the regime where $M/\Lambda \simeq \zeta \sim 0.1 \div 0.3$, at least, albeit not much less. In this case, the experimental bound $k > 10\ \mathrm{mm}^{-1} = 2\cdot 10^{-3}$ eV [63] turns out to be compatible with the lower bound [33, 39] for the localization scale $M > (2 \div 3)$ TeV, which makes somewhat challenging to produce new physics related to the fifth dimension at the next generation of colliders. The corresponding cut-off is $\Lambda > 10 \div 20$ TeV so that, from eq. (5.9), perturbation theory is controlled by the very tiny constant $\overline{\kappa} > 10^{-15}$, whilst the five dimensional cosmological constant must be tuned to the value $\lambda \sim 10^{-2}\ \mathrm{MeV}^2$.

Let us find the relationship between the bare value $\mathcal{G}$ and dressed value $\kappa\mathcal{G}$ of the five dimensional gravitational constants. According to eq. (3.6), the latter one is controlled by the ratio

$$\omega = \frac{N\Lambda^2\mathcal{G}}{54\pi^3} = \frac{\overline{\kappa}\,\pi^6}{9N^2\zeta^2} \sim 10\,\overline{\kappa} \sim 10^{-14} \ll 1 \tag{5.11}$$

for the chosen values of parameters $\overline{\kappa}$ and ζ. Thus we conclude that $\kappa \simeq 1$ and therefore the bare gravitational constant $\mathcal{G}$ mostly determines the intensity of the gravitational attraction in the five dimensional space-time. But this gravitational force is pretty strong, as its Planck scale $M_* \sim 10^8$ GeV is much lower than its four dimensional counterpart $M_P \sim 10^{19}$ GeV.

A further economical choice might be to identify the AdS curvature scale k with the electroweak symmetry breaking scale $\mu \sim 200$ GeV with the hope to connect the top-quark mass formation in the Standard Model to some extra dimensional gravitational effects. If $k \sim \mu \sim 200$ GeV, and still $\zeta \sim 0.1$ then one finds $M \sim 10^{10}$ GeV and $\Lambda \sim 10^{11}$ GeV so that the five dimensional cosmological constant must be tuned to $\lambda \sim 10^{12}\ \mathrm{GeV}^2$.

It follows therefrom that the expansion parameter increases up to $\overline{\kappa} \sim 10^{-8}$. Nevertheless, it does keep to be very small and, consequently, one basically needs the first orders of perturbation theory in the gravitational interaction to the aim of reaching a sufficiently good precision. For such a choice, the only signatures of extra dimensional physics could come from branon detection and there is no hope to reach sufficiently high energies to overcome the barrier M towards the fifth dimension. The estimates coming from eqs. (5.11) and (5.10), *i.e.* $\omega \sim 10^{-8} \ll 1\,; \quad M_* \sim 10^{13}$ GeV, show that this scenario still corresponds to the fundamental five dimensional gravity Action with $\kappa \simeq 1$, a relatively strong gravity.

The second scenario under consideration is that one of the `induced gravity`. Let us adopt the induced gravity relations of eq. (3.8)

$$\kappa \simeq 54\pi^3/N\Lambda^2\mathcal{G} \; ; \qquad \overline{\kappa}^{1/2} \simeq 3M/\Lambda = 3N\zeta/\pi^3 \ll 1 \; , \tag{5.12}$$

in such a way that

$$kM_P^2 = 4N\Lambda^3/27\pi^3 \; ; \qquad k^5 M_P^4 = 128N^2M^9/27\pi^6 \; . \tag{5.13}$$

In particular, for a lower experimental bound $k \geq 2 \cdot 10^{-12}$ GeV one finds

$$\begin{aligned} &M \geq 100 \text{ GeV} \; , \qquad \Lambda \geq 10^9 \text{ GeV} \; ; \\ &\zeta \sim M/\Lambda \sim 10^{-7} \; , \qquad \overline{\kappa} \sim 10^{-13} \; . \end{aligned} \tag{5.14}$$

This means that the light particle interaction is highly suppressed and the only particle interaction which is left is the gravitational one. It is worthwhile to remark that the induced gravity still keeps to be weak at low energies $\overline{\kappa} \ll 1$, which also entails the equivalence between the compositeness scale Λ_c defined in eq. (3.3) and the five dimensional Planck-like scale $M_* > 10^8$ GeV (see eq. (5.10)). Evidently the barrier $M \sim 100$ GeV is too low to be accepted by modern collider experiments, as the gravity (and gauge bosons) may easily be able to give fermions enough energy to disappear from our world.

For so far experimentally acceptable barriers of order $M \sim 1$ TeV, one finds that the AdS curvature scale $k \sim 10^{-10}$ GeV, which corresponds to distances of the μm order, becomes certainly unreachable in a nearest future experiment hunting for Newton's law deviations [63]. As a matter of fact, the scalar particles essentially decouple from the fermion world and from each other since $\Lambda_c \simeq M_* \sim 10^9$ GeV and $\zeta \sim M/\Lambda \sim 10^{-6.5}$; $\overline{\kappa} \sim 10^{-12}$. Although the Higgs-like particles may be involved into the gauge boson interaction and be observable by gauge boson mediation, it turns out that branons, *i.e.* the quanta of the field ϕ, do represent excitations of a geometrical nature, being related to the Goldstone bosons of the translational invariance breaking. As a consequence, their actual decoupling from any other kind of matter makes them a perfect candidate for the dark matter/energy, depending on their mass.

For the electroweak breaking scale $k \sim 2 \cdot 100$ GeV, the barrier $M \sim 10^{10}$ GeV is too high to trigger any observable physics at the Earth laboratories since $\Lambda \sim 10^{14}$GeV ; $\zeta \sim M/\Lambda \sim 10^{-4}$, whereas branons keep essentially decoupled and thereby belonging to the dark universe. The low-energy gravity remains weak with $\overline{\kappa} \sim 10^{-8}$ so that we see that, for matter induced gravity, the parameter ζ may not be small only for scales $k \sim M \sim \Lambda$ approaching the Planck scale, where gravity is actually strong ($\overline{\kappa} \sim 1$) and quantum gravity is in order.

6. Conclusions, further development

In this paper we have performed the systematic analysis of how the brane world can be generated dynamically by matter self-interaction from a spontaneous breaking of the translational invariance. The latter breaking is intimately related to the τ-symmetry breaking

in our proto-fermion model. The pertinent low-energy effective Action for fermions and the auxiliary scalar fields, as well as for their gravitational interaction, has been obtained by one-loop integration of high energy spinor degrees of freedom.

Both the gravitational and the scalar fields turn out to be responsible for the localization of light matter on the brane. Nevertheless, even if the scalar kink-like vacuum solution allows to trap light fermion and scalar particles inside the brane layer, still the AdS_5 gravitational background, in which the kink-like vacuum configuration are embedded, just induces the quantum tunneling taking away, in principle, some of those particles. Meanwhile, some simple WKB estimations prove that such a decay, eventually due to the quantum tunnel effect, is extremely well suppressed, so that it becomes practically impossible to detect the disappearance of any particle in the visible universe during the universe lifetime.

These circumstances have allowed us to formulate the following principle:
particle lifetimes on the brane must be evaluated by wave packet localization, which is unambiguously and consistently determined in perturbation theory. Within this approach, the mass spectrum of light states on the brane turns out to be slightly different from the corresponding one in the flat space limit.

We have in turn examined the possible ranges of different scales and coupling constants in terms of the Newton's constant normalization to its observed value and in terms of the experimental bounds on the AdS_5 curvature and thresholds on appearance/disappearance of high energy particles in accelerator experiments. As a summary of these studies, we conclude that:
a) on the one hand, fundamental gravity in five dimensions appears to be more challenging for future experiments, because the phenomenologically acceptable large values of AdS_5 curvature $k \sim 10^{-3}$ eV together with the relatively low translational symmetry breaking scales $M \sim 1 \div 3$ TeV turn out to be compatible with a weak, but not vanishing, coupling of branons to Higgs-like scalars and fermions. However, it appears that the lower are the values of the AdS_5 curvature k, the higher is the threshold for new physics M and the weaker is the interaction among spinor and scalar matter, in such a way to move branons to the dark side of the universe.
b) On the other hand, induced gravity leads to decoupling of branons from other matter in a wide range of acceptable scales and coupling constants, thus putting them straightforwardly to the dark matter realm.
c) In any case, the dimensionless parameter which characterizes the strength of the gravitational interaction is very small, of the order $\overline{\kappa} \leq 10^{-8}$. This feature does justify the use of the perturbation theory both in the calculation of vacuum field configurations and gravity background, as well as in the derivation of mass spectrum of localized particles.

For nearest future developments, we deserve the following interesting problems:
a) interplay in deviation from the scaling point (where only zero-mass states are localized on the brane) between manifest driving with the scale μ and mass generation by the AdS background gravity;
b) next-to-leading effects in the gravitational coupling constant $\bar{\kappa}$ and, in particular, scalar gravity ("radion"[81, 82, 83]) influence on the matter spectrum;

c) soft breaking of the translational symmetry due to scalar field defects, necessarily accompanied with defects in the cosmological constant and consequent production of massive branons;
d) supplementing our model with gauge bosons;
e) search for new experimental perspectives to detect branons and other signatures of extra dimensions based on the model presented in our paper.

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A. One-loop effective Action

Let us consider the second order matrix-valued positive elliptic differential operator

$$\not{\mathbf{D}}^\dagger \not{\mathbf{D}} = -\, g^{AB}(X) D_A D_B + \mathcal{M}^2(X) \ ,$$
$$\mathcal{M}^2(X) \equiv \frac{1}{4} R(X) + \Phi^2(X) + H^2(X) - \tau_3\, \not{\partial}\Phi(X) - \tau_1\, \not{\partial} H(X) \ , \tag{A.1}$$

acting on a n-dimensional Riemannian manifold (eventually $n = 5$). Our aim is to evaluate the matrix element of the distribution of the states operator: namely,

$$\langle X|\vartheta(Q^2 - \not{\mathbf{D}}^\dagger \not{\mathbf{D}})|Y\rangle = \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i}\, \frac{\exp\{tQ^2\}}{t}\, \langle X|\exp\{-t\, \not{\mathbf{D}}^\dagger \not{\mathbf{D}}\}|Y\rangle \ , \quad c > 0 \ . \tag{A.2}$$

As a matter of fact, if the positive operator $\not{\mathbf{D}}^\dagger \not{\mathbf{D}}$ is also of the trace class, then the trace of $\vartheta(Q^2 - \not{\mathbf{D}}^\dagger \not{\mathbf{D}})$ does represent the number of the eigenstates of $\not{\mathbf{D}}^\dagger \not{\mathbf{D}}$ up to the momentum square Q^2. It is convenient to write the heat kernel in the form

$$\langle X|\exp\{-t\, \not{\mathbf{D}}^\dagger \not{\mathbf{D}}\}|Y\rangle \equiv (4\pi t)^{-n/2} \exp\left\{-\frac{(X-Y)^2}{4t}\right\} \Omega(t|X,Y) \ , \tag{A.3}$$

where $\Omega(t|X,Y)$ is the so called transport function which fulfills

$$\left(\partial_t + \frac{X\cdot\partial}{t} + \not{\mathbf{D}}_X^\dagger \not{\mathbf{D}}_X\right) \Omega(t|X,Y) = 0 \ , \tag{A.4}$$

$$\lim_{t\downarrow 0} \Omega(t|X,Y) = \widehat{\mathbf{1}} \ , \tag{A.5}$$

in such a way that

$$\lim_{t\downarrow 0}\ \langle X|\exp\{-t\, \not{\mathbf{D}}^\dagger \not{\mathbf{D}}\}|Y\rangle = \widehat{\mathbf{1}}\delta^{(n)}(X-Y) \ . \tag{A.6}$$

If we insert eq. (A.3) in eq. (A.2) and change the integration variable we obtain

$$\langle X|\vartheta(Q^2 - \not{\mathbf{D}}^\dagger \not{\mathbf{D}})|Y\rangle = 2Q^n (4\pi)^{-1-n/2} \int_{c-i\infty}^{c+i\infty} dt\ t^{-1-n/2}$$
$$\times\ \exp\left\{t - \frac{Q^2(X-Y)^2}{4t} - \frac{i\pi}{2}\right\} \Omega\left(\frac{t}{Q^2}\,|X,Y\right) \ . \tag{A.7}$$

Now, if we write

$$\Omega\left(\frac{t}{Q^2}\,|X,Y\right) = \sum_{k=0}^{[n/2]} t^k a_k(X,Y) Q^{-2k} + \mathcal{R}_{[n/2]+1}\left(\frac{t}{Q^2}\,|X,Y\right)\ , \tag{A.8}$$

it turns out that, by insertion of eq. (A.8) into the integral (A.7), the last term in the RHS of the above expression becomes sub-leading and negligible in the limit of very large Q. As a consequence, the leading asymptotic behavior of the matrix element (A.2) in the large Q limit reads [84]

$$\begin{aligned}
\langle X|\vartheta(Q^2 - \not{\mathbf{D}}^\dagger \not{\mathbf{D}})|Y\rangle \ &\overset{Q\to\infty}{\sim}\ (4\pi)^{-n/2} \sum_{k=0}^{[n/2]} a_k(X,Y) Q^{n-2k} \\
&\qquad \times \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i}\, t^{k-1-n/2} \exp\left\{t - \frac{Q^2(X-Y)^2}{4t}\right\} \\
&\overset{X\to Y}{\sim}\ (4\pi)^{-n/2} \sum_{k=0}^{[n/2]} a_k(X,Y)\, \frac{Q^{n-2k}}{\Gamma(1-k+n/2)}\ .
\end{aligned} \tag{A.9}$$

In the diagonal limit $X = Y$, for $n = 4, 5$ the relevant coefficients of the heat kernel asymptotic expansion take the form [75]

$$\begin{aligned}
a_0(X,X) &\equiv \widehat{\mathbf{1}}\ ; \qquad a_1(X,X) = \frac{R(X)}{6} - \mathcal{M}^2(X)\ ; \\
a_2(X,X) &= \frac{1}{2}[\, a_1(X,X)\,]^2 - \frac{1}{6}\, D^2\mathcal{M}^2(X) + \frac{1}{30}\, D^2 R(X) \\
&+ \frac{1}{180}\left[\, R^{ABCD}(X) R_{ABCD}(X) - R^{AB}(X) R_{AB}(X)\,\right] \\
&+ \frac{1}{12}\, \Omega^{AB}(X)\Omega_{AB}(X)\ ,
\end{aligned} \tag{A.10}$$

where the curvature bundle, *i.e.* the field strength associated to the spin connection, reads

$$\Omega_{AB}(X) \equiv \frac{1}{8}\, [\,\widehat{\gamma}_j, \widehat{\gamma}_k\,]\, R_{ABjk}(X)\ . \tag{A.11}$$

In such a way, we can write down the dominant diagonal matrix element, leading eventually to the five dimensional Euclidean low–energy Lagrange density, in the form

$$\begin{aligned}
&\langle X|\vartheta(Q^2 - D^\dagger D)|X\rangle \overset{Q\to\infty}{\sim} \\
&\frac{Q^5}{60\pi^3}\left\{\widehat{\mathbf{1}} + \frac{5}{2}\, Q^{-2}\, a_1(X,X) + \frac{15}{4}\, Q^{-4}\, a_2(X,X)\right\}\ .
\end{aligned} \tag{A.12}$$

Furthermore we find $\quad [\ \mathrm{tr}\, a_0(X,X) = \mathrm{tr}\,\widehat{\mathbf{1}} = 8\ ]$

$$\frac{1}{8}\,\mathrm{tr}\, a_1(X,X) = -\frac{1}{12}\, R(X) - [\,\Phi^2(X) + H^2(X)\,]\ ; \tag{A.13}$$

$$\begin{aligned}
\frac{1}{8}\,\mathrm{tr}\,[\, a_1(X,X)\,]^2 &= \frac{1}{144}\, R^2(X) + \frac{1}{6}\, R(X)[\,\Phi^2(X) + H^2(X)\,] \\
&+ \partial_A\Phi(X)\partial^A\Phi(X) + \partial_A H(X)\partial^A H(X) \\
&+ [\,\Phi^2(X) + H^2(X)\,]^2
\end{aligned} \tag{A.14}$$

so that we eventually obtain

$$
\begin{aligned}
\frac{1}{8}\,\mathrm{tr}\,a_2(X,X) \;=\; & \frac{1}{2}\,\partial_A\Phi(X)\partial^A\Phi(X)+\frac{1}{2}\,\partial_A H(X)\partial^A H(X) \\
& +\frac{1}{2}[\,\Phi^2(X)+H^2(X)\,]^2+\frac{1}{12}\,R(X)[\,\Phi^2(X)+H^2(X)\,] \\
& -\frac{1}{6}\,D^2[\,\Phi^2(X)+H^2(X)\,]-\frac{1}{120}\,D^2R(X)+\frac{1}{288}\,R^2(X) \\
& -\frac{1}{180}R^{AB}(X)R_{AB}(X)-\frac{7}{1440}\,R^{ABCD}(X)R_{ABCD}(X)\;. \qquad \text{(A.15)}
\end{aligned}
$$

B. Curvature tensors for conformal metric

Consider the five dimensional Riemannian metric

$$
\begin{aligned}
& g_{\mu\nu}(x,z)=g_{\mu\nu}(x)\,\exp\{-2\rho(z)\}\;, \\
& g_{\mu 5}(x,z) \;=\; g_{5\nu}(x,z) \;=\; 0\;, \\
& g_{55}(x,z) \;=\; 1\;. \qquad \text{(B.1)}
\end{aligned}
$$

The related Christoffel symbols take the values

$$
\begin{aligned}
& \Gamma^{\lambda}_{\mu\nu}(x,z)=\frac{1}{2}\,g^{\lambda\kappa}(x)\,\{\partial_\mu g_{\nu\kappa}(x)+\partial_\nu g_{\mu\kappa}(x)-\partial_\kappa g_{\mu\nu}(x)\}\equiv\Gamma^{\lambda}_{\mu\nu}(x)\;, \\
& \Gamma^{\lambda}_{\mu 5}(x,z) \;=\; \Gamma^{\lambda}_{5\mu}(x,z) \;=\; -\,\rho'(z)\,\delta^{\lambda}_{\mu} \;\equiv\; \Gamma^{\lambda}_{\mu 5}(z)\;, \\
& \Gamma^{5}_{\mu\nu}(x,z) \;=\; \rho'(z)\,\exp\{-2\rho(z)\}\,g_{\mu\nu}(x)\;, \\
& \Gamma^{5}_{5\mu}(x,z) \;=\; \Gamma^{\lambda}_{55}(x,z) \;=\; \Gamma^{5}_{55}(x,z) \;=\; 0\;. \qquad \text{(B.2)}
\end{aligned}
$$

The corresponding Riemann tensor has components

$$
\begin{aligned}
& R^{\alpha}{}_{\beta\mu\nu}(x,z) \;=\; R^{\alpha}{}_{\beta\mu\nu}(x) \;+\; [\,\rho'(z)\,]^2\,\exp\{-2\rho(z)\}\,\left\{g_{\beta\mu}(x)\,\delta^{\alpha}_{\nu}-g_{\beta\nu}(x)\,\delta^{\alpha}_{\mu}\right\}\;, \\
& R_{5\beta 5\nu}(x,z) \;=\; \exp\{-2\rho(z)\}\,g_{\beta\nu}(x)\,\left\{\rho''(z)-[\,\rho'(z)\,]^2\right\}\;, \\
& R^{\alpha}{}_{5\mu\nu}(x,z) \;=\; R^{\alpha}{}_{\beta 5\nu}(x,z) \;=\; R_{5\beta\mu\nu}(x,z) \;=\; 0\;. \qquad \text{(B.3)}
\end{aligned}
$$

The components of the Ricci tensor read

$$
\begin{aligned}
& R_{\beta\nu}(x,z) \;=\; R_{\beta\nu}(x)+\exp\{-2\rho(z)\}\,g_{\beta\nu}(x)\,\left\{\rho''(z)-4[\,\rho'(z)\,]^2\right\}\;, \\
& R_{55}(x,z) \;=\; 4\left\{\rho''(z)-[\,\rho'(z)\,]^2\right\}\;, \\
& R_{\beta 5}(x,z) \;=\; 0\;, \qquad \text{(B.4)}
\end{aligned}
$$

so that the scalar curvature becomes

$$
R(x,z)=\exp\{2\rho(z)\}\,R(x)+8\rho''(z)-20[\,\rho'(z)\,]^2\;. \qquad \text{(B.5)}
$$

It is also useful to compute the quadratic invariant in the Riemann tensor: we find

$$
\begin{aligned}
& R_{ABCD}(x,z)\,R^{ABCD}(x,z)= \\
& R_{\alpha\beta\mu\nu}(x,z)\,R^{\alpha\beta\mu\nu}(x,z)+4R_{\alpha 5\mu 5}(x,z)R^{\alpha 5\mu 5}(x,z)\;. \qquad \text{(B.6)}
\end{aligned}
$$

We have

$$
\begin{aligned}
R^{\alpha\beta\mu\nu}(x,z) &= \exp\{6\rho(z)\}\, R^{\alpha\beta\mu\nu}(x) \\
&\quad + [\,\rho'(z)\,]^2 \exp\{4\rho(z)\} \left\{ g^{\alpha\nu}(x) g^{\beta\mu}(x) - g^{\alpha\mu}(x) g^{\beta\nu}(x) \right\} \;, \\
R_{\alpha\beta\mu\nu}(x,z) &= \exp\{-2\rho(z)\}\, R_{\alpha\beta\mu\nu}(x) \\
&\quad + [\,\rho'(z)\,]^2 \exp\{-4\rho(z)\} \;\{ g_{\alpha\nu}(x) g_{\beta\mu}(x) - g_{\alpha\mu}(x) g_{\beta\nu}(x) \} \;,
\end{aligned}
\tag{B.7}
$$

so that

$$
\begin{aligned}
R_{\alpha\beta\mu\nu}(x,z)\, R^{\alpha\beta\mu\nu}(x,z) &= \exp\{4\rho(z)\}\, R_{\alpha\beta\mu\nu}(x)\, R^{\alpha\beta\mu\nu}(x) \\
&\quad - 4\exp\{2\rho(z)\}\, [\,\rho'(z)\,]^2 R(x) + 24[\,\rho'(z)\,]^4
\end{aligned}
\tag{B.8}
$$

and finally

$$
\begin{aligned}
R_{ABCD}(x,z)\, R^{ABCD}(x,z) &= 16[\,\rho''(z)\,]^2 - 32\rho''(z)[\,\rho'(z)\,]^2 + 40[\,\rho'(z)\,]^4 \\
&\quad + \exp\{4\rho(z)\}\, R_{\alpha\beta\mu\nu}(x)\, R^{\alpha\beta\mu\nu}(x) \\
&\quad - 4\exp\{2\rho(z)\}\, [\,\rho'(z)\,]^2 R(x) \;.
\end{aligned}
\tag{B.9}
$$

In a similar way we obtain the quadratic invariant in the Ricci tensor that reads

$$
\begin{aligned}
R_{AB}(x,z)\, R^{AB}(x,z) &= R_{\alpha\beta}(x,z)\, R^{\alpha\beta}(x,z) + R_{55}(x,z) R^{55}(x,z) \\
&= 20[\,\rho''(z)\,]^2 - 64\rho''(z)[\,\rho'(z)\,]^2 + 80[\,\rho'(z)\,]^4 \\
&\quad + 2\exp\{2\rho(z)\}\, R(x) \left\{ \rho''(z) - 4[\,\rho'(z)\,]^2 \right\} \\
&\quad + \exp\{4\rho(z)\}\, R_{\alpha\beta}(x)\, R^{\alpha\beta}(x) \;.
\end{aligned}
\tag{B.10}
$$

Finally, we easily get the quadratic invariant in the curvature scalar

$$
\begin{aligned}
R^2(x,z) &= 64[\,\rho''(z)\,]^2 + 400[\,\rho'(z)\,]^4 - 320\rho''(z)[\,\rho'(z)\,]^2 \\
&\quad + 8\exp\{2\rho(z)\}\, R(x) \left\{ 2\rho''(z) - 5[\,\rho'(z)\,]^2 \right\} \\
&\quad + \exp\{4\rho(z)\}\, R^2(x) \;.
\end{aligned}
\tag{B.11}
$$

Consider now the special case of a *quasi–flat* Riemannian metric

$$
\begin{aligned}
g_{\mu\nu}(z) &= \delta_{\mu\nu}\, \exp\{-2\rho(z)\} \;, \\
g_{\mu 5}(z) &= g_{5\nu}(z) = 0 \;, \\
g_{55}(z) &= 1 \;.
\end{aligned}
\tag{B.12}
$$

The related Christoffel symbols take the values

$$
\begin{aligned}
\Gamma^{\lambda}_{\mu\nu}(z) &= 0 \;, \\
\Gamma^{\lambda}_{\mu 5}(z) &= \Gamma^{\lambda}_{5\mu}(z) = -\,\rho'(z)\, \delta^{\lambda}_{\mu} \;, \\
\Gamma^{5}_{\mu\nu}(z) &= \rho'(z)\, \exp\{-2\rho(z)\}\, \delta_{\mu\nu} \;, \\
\Gamma^{5}_{5\mu}(z) &= \Gamma^{\lambda}_{55}(z) = \Gamma^{5}_{55}(z) = 0 \;.
\end{aligned}
\tag{B.13}
$$

where $\rho'(z) \equiv (d\rho/dz)$. The corresponding Riemann tensor has components

$$
\begin{aligned}
R^{\alpha}{}_{\beta\mu\nu}(z) &= [\,\rho'(z)\,]^2 \exp\{-2\rho(z)\} \left\{\delta_{\beta\mu}\,\delta^{\alpha}_{\nu} - \delta_{\beta\nu}\,\delta^{\alpha}_{\mu}\right\} \;, \\
R_{5\beta5\nu}(z) &= \exp\{-2\rho(z)\}\,\delta_{\beta\nu}\left\{\rho''(z) - [\,\rho'(z)\,]^2\right\} \;, \\
R^{\alpha}{}_{5\mu\nu}(z) &= R^{\alpha}{}_{\beta5\nu}(z) \;=\; R_{5\beta\mu\nu}(z) \;=\; 0 \;.
\end{aligned}
\tag{B.14}
$$

The components of the Ricci tensor read

$$
\begin{aligned}
R_{\beta\nu}(z) &= \exp\{-2\rho(z)\}\,\delta_{\beta\nu}\left\{\rho''(z) - 4[\,\rho'(z)\,]^2\right\} \;, \\
R_{55}(z) &= 4\left\{\rho''(z) - [\,\rho'(z)\,]^2\right\} \;, \\
R_{\beta5}(z) &= 0 \;,
\end{aligned}
\tag{B.15}
$$

so that the scalar curvature becomes

$$
R(z) = 8\rho''(z) - 20[\,\rho'(z)\,]^2 \;. \tag{B.16}
$$

The corresponding quadratic invariants in the Riemann tensor, the Ricci tensor and the curvature scalar become

$$
R_{ABCD}(z)\,R^{ABCD}(z) = 16[\,\rho''(z)\,]^2 - 32\rho''(z)[\,\rho'(z)\,]^2 + 40[\,\rho'(z)\,]^4 \tag{B.17}
$$

$$
R_{AB}(z)\,R^{AB}(z) = 20[\,\rho''(z)\,]^2 - 64\rho''(z)[\,\rho'(z)\,]^2 + 80[\,\rho'(z)\,]^4 \tag{B.18}
$$

$$
R^2(z) = 64[\,\rho''(z)\,]^2 - 320\rho''(z)[\,\rho'(z)\,]^2 + 400[\,\rho'(z)\,]^4 \;. \tag{B.19}
$$

Finally, for a scalar function $f(z)$ of the fifth coordinate we have

$$
D^C\partial_C f(z) \;=\; f''(z) - 4\rho'(z)\,f'(z) \;; \tag{B.20}
$$

$$
\begin{aligned}
&\left\{g_{\alpha\alpha}(z)D^C\partial_C - D_\alpha\partial_\alpha\right\} f(z) \;= \\
&\exp\{-2\rho(z)\}\left\{f''(z) - 3\rho'(z)\,f'(z)\right\} \;;
\end{aligned}
\tag{B.21}
$$

$$
\left(D^C\partial_C - D_5\partial_5\right) f(z) \;=\; -4\rho'(z)\,f'(z) \;. \tag{B.22}
$$

C. Equations of motion in conformal metric

In the case of conformally flat metric (3.10) and for a kink-like pair of scalar fields

$$
\langle\Phi(X)\rangle_0 = \Phi(z) \;, \qquad \langle H(X)\rangle_0 = H(z) \;, \tag{C.1}
$$

from the full low-energy Lagrange density (3.1) one finds the following gravitational field equations in which the quadratic terms in the curvature tensors are suitably taken into account: namely,

$$
\begin{aligned}
&\rho''(z)\,(\delta_{A5}\,\delta_{B5} - 1) + \left[\,2\rho'^2(z) + \lambda/3\,\right]\delta_{AB} \;= \\
&= \;(N\kappa\mathcal{G}/12\pi^3)\left\{(2\delta_{A5}\,\delta_{B5} - \delta_{AB})\left[\,\Phi'^2(z) + H'^2(z)\,\right]\right. \\
&+ \;\delta_{AB}\,\left(2\Delta_1\Phi^2(z) + 2\Delta_2 H^2(z) - \left[\,\Phi^2(z) + H^2(z)\,\right]^2\right) \\
&+ \;\left[\,\rho''(z)\,(\delta_{A5}\,\delta_{B5} - \delta_{AB}) + 2\rho'^2(z)\delta_{AB}\,\right]\left[\,\Phi'^2(z) + H'^2(z)\,\right] \\
&+ \;(1/3)\,\left(\delta_{AB}\left[\,\partial_z^2 - 3\rho'(z)\,\partial_z\,\right] - \delta_{A5}\,\delta_{B5}\left[\,\partial_z^2 + \rho'(z)\,\partial_z\,\right]\right) \;\times \\
&\times\;\left.\left[\,\Phi'^2(z) + H'^2(z)\,\right] - \;\delta_{AB}\;F_1(R^2) \;+\; \delta_{A5}\,\delta_{B5}\;F_2(R^2)\right\} \;,
\end{aligned}
\tag{C.2}
$$

supplemented with eqs. (3.20) for matter fields. Here the higher order terms $F_{1,2}(R^2)$ can be calculated from that part of the low-energy Euclidean Action (3.1), which is quadratic in the curvature tensors, as we shall see below. These equations can be derived from the effective Action with Lagrange density (3.11) by means of two types of variations. The variation with respect to the conformal factor $\rho(z)$ gives rise to the contribution into the partial trace of eq.s (C.2), whereas the infinitesimal change of the z-coordinate, $dz' = dz[\,1-\epsilon(z)\,]$ drives to the variation of the $g_{55}(z)$ component of the metric (3.10). The latter equation can be derived directly from eq.s (3.11) according to the following rules: namely,

$$\begin{aligned} \delta g_{55}(z) \ &= -\ 2\epsilon(z)\ ; \qquad \delta\rho'(z) \ = \ \rho'(z)\,\epsilon(z)\ ; \\ \delta\rho''(z) \ &= 2\rho''(z)\,\epsilon(z) \ + \ \rho'(z)\,\epsilon'(z)\ . \end{aligned} \tag{C.3}$$

Now, from the equality

$$\begin{aligned} &\frac{N\Lambda}{2880\pi^3}\int d^5X\sqrt{g}\,\{5R^2(X) - 8R_{AB}(X)R^{AB}(X) - 7R_{ABCD}(X)R^{ABCD}(X)\} \\ &= \ \frac{N\Lambda}{120\pi^3}\int d^5X\ \ \exp\{-4\rho(z)\}\,\{\,2[\,\rho''(z)\,]^2 - 36\rho''(z)[\,\rho'(z)\,]^2 + 45[\,\rho'(z)\,]^4\,\} \end{aligned}$$

and by means of the above mentioned two kinds of variations, a straightforward calculation shows that the additional contribution to the equations of motion can be represented in terms of the two scalar functions

$$\begin{aligned} F_1(R^2) &= (1/30)\ \left[\,9(\rho')^4 + 6(\rho'')^4 + 8\rho'\rho''' - 25\rho''(\rho')^2 - \rho''''\,\right]\ ; \\ F_2(R^2) &= (1/30)\ \left[\,8(\rho'')^2 + 4\rho'\rho''' - 9\rho''(\rho')^2 - \rho''''\,\right]\ . \end{aligned} \tag{C.4}$$

On the one hand, it is apparent that the difference between the fifth and any other component of eq.s (C.2) does not include the cosmological constant λ, since we have

$$\rho'' = \frac{\overline{\kappa}}{M^2}\left\{\Phi'^2 + H'^2 + \frac{1}{2}\left(\rho'' - \frac{1}{3}\frac{d^2}{dz^2} - \frac{1}{3}\rho'\frac{d}{dz}\right)(\Phi^2 + H^2) + \frac{1}{2}F_2(R^2)\right\} \tag{C.5}$$

with $\overline{\kappa}$ being defined in eq. (3.22). On the other hand – compare with eq. (3.34) – the fifth component actually represents the integral of motion for the remaining three equations (3.20) and (C.5), in which λ plays the role of an integration constant

$$\begin{aligned} 2M^2\lambda_{\text{eff}} \ &= \Phi'^2 + H'^2 + 2\Delta_1\Phi^2 + 2\Delta_2 H^2 - \left(\Phi^2 + H^2\right)^2 \\ &- \ \frac{4M^2}{\overline{\kappa}}\,\rho'^2 + \left(2\rho'^2 - \frac{4}{3}\rho'\frac{d}{dz}\right)(\Phi^2 + H^2) \\ &+ \ F_2(R^2) - F_1(R^2)\ , \end{aligned} \tag{C.6}$$

where the definition for λ_{eff} is adopted from eq.(3.23). As a matter of fact, the very last statement can be proved by differentiation of the above relationship and substitution of eq.s (3.20) and (C.5). Moreover, the use of the following identity turns out to be crucial: namely,

$$4\rho' F_2(R^2) = \frac{d}{dz}\ \left[\,F_2(R^2) - F_1(R^2)\,\right]\ . \tag{C.7}$$

The latter result is a direct consequence of the general covariance of the equations of motion.

D. Gravitational field equation: general metric

Our starting point is the low energy effective Action involving scalar and gravitational fields: namely,

$$
\begin{aligned}
S_{\rm eff}(\Phi, H, g) \;\equiv\; & \frac{N\Lambda}{4\pi^3} \int d^5X \;\sqrt{g(X)}\;\Big\{ -\,\frac{2\pi^3}{N\kappa\mathcal{G}}\,R(X) + \frac{4\pi^3}{N\kappa\mathcal{G}}\,\lambda \\
& + \partial_A\Phi(X)\partial^A\Phi(X) + \partial_A H(X)\partial^A H(X) - 2\Delta_1\Phi^2(X) - 2\Delta_2 H^2(X) \\
& + \;\big[\,\Phi^2(X) + H^2(X)\,\big]^2 + \frac{1}{6}\,R(X)\;\big[\,\Phi^2(X) + H^2(X)\,\big]\Big\}
\end{aligned} \tag{D.1}
$$

The first variation δg^{AB} with respect to the metric leads to the equations of motion for the gravitational field. Taking into account the identities (see [85], p.453)

$$
\delta\sqrt{g} \;=\; -\;\frac{1}{2}\,\sqrt{g}\,g_{AB}\,\delta g^{AB}\;, \tag{D.2}
$$

$$
g^{AB}\,\delta R_{AB} \;=\; -\;D_A\left(D_B\,\delta g^{AB} \;+\; g^{CD}\,D^A\,\delta g_{CD}\right)\;, \tag{D.3}
$$

we find

$$
\begin{aligned}
\delta S_{\rm eff}[\,\Phi, H, g\,] = -\;& \frac{N\Lambda}{8\pi^3}\int d^5X\sqrt{g}\,\{\,\partial_C\Phi(X)\partial^C\Phi(X) + \partial_C H(X)\partial^C H(X) \\
-\;& 2\Delta_1\Phi^2(X) - 2\Delta_2 H^2(X) + [\,\Phi^2(X) + H^2(X)\,]^2 \\
+\;& \frac{4\pi^3\lambda}{N\kappa\mathcal{G}} + \frac{R(X)}{6}\left[\Phi^2(X) + H^2(X) - \frac{12\pi^3}{N\kappa\mathcal{G}}\right]\Big\}\,g_{AB}\,\delta g^{AB} \\
+\;& \frac{N\Lambda}{4\pi^3}\int d^5X\sqrt{g}\;\;\{\,\partial_A\Phi(X)\partial_B\Phi(X) \;+\partial_A H(X)\partial_B H(X) \\
+\;& \frac{1}{6}\;R_{AB}(X)\,[\,\Phi^2(X) + H^2(X) - 12\pi^3/N\kappa\mathcal{G}\,]\Big\}\,\delta g^{AB} \\
-\;& \frac{N\Lambda}{24\pi^3}\int d^5X\sqrt{g}\;[\,\Phi^2(X) + H^2(X) + 12\pi^3/N\kappa\mathcal{G}\,] \\
\times\;& D_A\left(D_B\,\delta g^{AB} \;+\; g^{CD}\,D^A\,\delta g_{CD}\right)\;.
\end{aligned} \tag{D.4}
$$

With the help of identity $\;\Gamma^A_{AB} = \partial_A \ln\sqrt{g}\;$ the very last integral can be calculated by parts yielding

$$
\begin{aligned}
& \frac{N\Lambda}{24\pi^3}\int d^5X\sqrt{g}\;[\,\Phi^2(X) + H^2(X) - 12\pi^3/N\kappa\mathcal{G}\,]\,D_A\left(D_B\,\delta g^{AB} \;+\; g^{CD}\,D^A\,\delta g_{CD}\right) \\
&= \frac{N\Lambda}{24\pi^3}\int d^5X\sqrt{g}\;\{D_B\,\partial_A\,[\,\Phi^2(X) + H^2(X)\,] -\; g_{AB}\,D^2\,[\,\Phi^2(X) + H^2(X)\,]\}\,\delta g^{AB} \\
& -\frac{N\Lambda}{24\pi^3}\int d^5X\;\partial_A\mathcal{B}^A(X)\;,
\end{aligned} \tag{D.5}
$$

where the boundary term takes the form

$$
\begin{aligned}
\mathcal{B}^A(X) \equiv\; & \sqrt{g}\,[\,\Phi^2(X) + H^2(X) - 12\pi^3/N\kappa\mathcal{G}\,] \\
& \times\;[\,D_B\,\delta g^{AB}(X) \;-\; g_{CD}(X)\,g^{AB}(X)\,D_B\,\delta g^{CD}(X)\,] \\
& -\;\sqrt{g}\,[\,\delta g^{AB}(X) - g^{AB}(X)\,g_{CD}(X)\,\delta g^{CD}(X)\,] \\
& \times\;\partial_B\,[\,\Phi^2(X) + H^2(X)\,]\;\;.
\end{aligned} \tag{D.6}
$$

so that we finally obtain, in the Euclidean case and up to the boundary term, the Einstein's equation in the presence of the cosmological term: namely,

$$R_{AB} - \frac{1}{2}\, g_{AB}\, (R - 2\lambda) \;=\; \frac{N\kappa\mathcal{G}}{2\pi^3}\, t_{AB} \tag{D.7}$$

where the normalized energy–momentum tensor of the scalar matter reads

$$\begin{aligned} t_{AB} \;&\equiv\; \partial_A\Phi\,\partial_B\Phi + \partial_A H\,\partial_B H \\ &-\; \frac{1}{2}\, g_{AB}\left[\,\partial_C\Phi\,\partial^C\Phi + \partial_C H\,\partial^C H - 2\Delta_1\Phi^2 - 2\Delta_2 H^2 + \left(\Phi^2 + H^2\right)^2\,\right] \\ &+\; \frac{1}{6}\left(R_{AB} - \frac{1}{2}\, g_{AB}\, R + g_{AB}\, D^C\partial_C - D_B\,\partial_A\right)\left(\Phi^2 + H^2\right)\ . \end{aligned} \tag{D.8}$$

References

[1] V.A. Rubakov, M.E. Shaposhnikov, *Phys. Lett.* **B 125** (1983) 136, 139 .

[2] K. Akama, Lect. Notes Phys. 176 (1982) 267 [hep-th/0001113];
M. Visser, *Phys. Lett.* **B 159** (1985) 22 [hep-th/9910093] ;
M. Pavsic, *Phys. Lett.* **A 116** (1986) 1 [gr-qc/0101075] ;
G.W. Gibbons, D.L. Wiltshire, *Nucl. Phys.* **B 287** (1987) 717 [hep-th/0109093] .

[3] N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, *Phys. Lett.* **B 429** (1998) 263 .

[4] A.Lukas, B.A. Ovrut, K.S. Stelle, D. Waldram,*Phys. Rev.* **D 59** (1999) 086001 .

[5] N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali,*Phys. Rev.* **D 59** (1999) 086004 .

[6] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, *Phys. Lett.* **B 436** (1998) 257.

[7] M. Gogberashvili, *Mod. Phys. Lett.* **A 14** (1999) 2025; *Int. J. Mod. Phys.* **D 11** (2002) 1639.

[8] L. Randall, R. Sundrum, *Phys. Rev. Lett.* **83** (1999) 3370, 4690.

[9] K.R. Dienes, E. Dudas, T. Gherghetta, *Phys. Lett.* **B 436** (1998) 55; *Nucl. Phys.* **B 537** (1999) 47;
Z. Berezhiani, I. Gogoladze, A. Kobakhidze, *Phys. Lett.* **B 522** (2001) 107.

[10] L.J. Hall, Y.Nomura, *Phys. Rev.* **D 64** (2001) 055003; *Phys. Rev.* **D 65** (2002) 125012; *Phys. Rev.* **D 66** (2002) 075004.

[11] I. Antoniadis, C. Muñoz, M. Quiros, *Nucl. Phys.* **B 397** (1993) 515;
I. Antoniadis, S. Dimopoulos, A. Pomarol, M. Quiros, *Nucl. Phys.* **B 544** (1999) 503;
I. Antoniadis, K. Benakli, M. Quiros, *Acta Phys. Polon.* **B33** (2002) 2477.

[12] E.A. Mirabelli, M.E. Peskin, *Phys. Rev.* **D 58** (1998) 065002;
Z. Kakushadze, S.H. Henry Tye, *Nucl. Phys.* **B 548** (1999) 180.

[13] H.-C. Cheng, B.A. Dobrescu, C.T. Hill, *Nucl. Phys.* **B 589** (2000) 249.

[14] R. Barbieri, L.J. Hall, Y. Nomura, *Phys. Rev.* **D 63** (2001) 105007;
R. Barbieri, L.J. Hall, G. Marandella, Y. Nomura, T. Okui, S.J. Oliver, M. Papucci, *Nucl. Phys.* **B 663** (2003) 141.

[15] A. Delgado, A. Pomarol, M. Quiros, *Phys. Rev.* **D 60** (1999) 095008; *J. High Energy Phys.* **01** (2000) 030.

[16] G. Altarelli, F. Feruglio, *Phys. Lett.* **B 511** (2001) 257.

[17] G.R. Dvali, M.A. Shifman, *Phys. Lett.* **B 475** (2000) 295.

[18] N. Arkani-Hamed, M. Schmaltz, *Phys. Rev.* **D 61** (2000) 033005;
H.C. Cheng, *Phys. Rev.* **D 60** (1999) 075015;
M. Bando, T. Kobayashi, T. Noguchi, K. Yoshioka, *Phys. Rev.* **D 63** (2001) 113017 ;
S. Chang, J. Hisano, H. Nakano, N. Okada, M. Yamaguchi, *Phys. Rev.* **D 62** (2000) 084025 ;
K. Fukazawa, T. Inagaki, Y. Katsuki, T. Muta, K. Ohkura, `hep-ph/0308022` .

[19] N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, N. Kaloper, *Phys. Rev. Lett.* **84** (2000) 586 ;
J. Lykken, L. Randall, *J. High Energy Phys.* **06** (2000) 014 ;
A. Chamblin, G.W. Gibbons, *Phys. Rev. Lett.* **84** (2000) 1090 .

[20] R. Gregory, V.A. Rubakov, S.M. Sibiryakov, *Phys. Rev. Lett.* **84** (2000) 5928 ;
Phys. Lett. **B 489** (2000) 203 .

[21] G.R. Dvali, G. Gabadadze, M. Porrati, *Phys. Lett.* **B 485** (2000) 208 ;
Mod. Phys. Lett. **A 15** (2000) 1717 .

[22] A. Albrecht, C.P. Burgess, F. Ravndal, C. Skordis, *Phys. Rev.* **D 65** (2002) 123507 .

[23] T. Han, J.D. Lykken, R.-J. Zhang, *Phys. Rev.* **D 59** (1999) 105006;
G.F. Giudice, R. Rattazzi, J.D. Wells, *Nucl. Phys.* **B 595** (2001) 250;

[24] S. Nussinov, R, Shrock,*Phys. Rev.* **D 59** (1999) 105002 ;
E.A. Mirabelli, M. Perelstein, M.E. Peskin, *Phys. Rev. Lett.* **82** (1999) 2236;
J.L. Hewett, *Phys. Rev. Lett.* **82** (1999) 4765 ;
H. Davoudiasl, J.L. Hewett, T.G. Rizzo, *Phys. Rev.* **D 63** (2001) 075004.

[25] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, J. March-Russell, *Phys. Rev.* **D 65** (2002) 024032;
G.R. Dvali, A.Y. Smirnov, *Nucl. Phys.* **B 563** (1999) 63;
Y. Grossman, M. Neubert, *Phys. Lett.* **B 474** (2000) 361.

[26] R. Barbieri, P. Creminelli, A. Strumia, *Nucl. Phys.* **B 585** (2000) 28.

[27] R. Gregory, V.A. Rubakov, S.M. Sibiryakov, *Class. Quan. Grav.* **17** (2000) 4437.

[28] S.L. Dubovsky, V.A. Rubakov, P.G. Tinyakov, *J. High Energy Phys.* **08** (2000) 041.

[29] A. Dobado, A.L. Maroto, *Nucl. Phys.* **B 592** (2001) 203;
J. Alcaraz, J.A.R. Cembranos, A. Dobado, A.L. Maroto, Phys. Rev. D **67**, 075010 (2003) ;
J.A.R. Cembranos, A. Dobado, A.L. Maroto, *Phys. Rev.* **D 68** (2003) 103505 ;
Phys. Rev. **D 70** (2004) 096001 .

[30] P. Creminelli, A. Strumia, *Nucl. Phys.* **B 596** (2001) 125 .

[31] V.A. Rubakov, *Sov. Phys. Usp.* **44** (2001) 871 ; *Sov. Phys. Usp.* **46** (2003) 211 .

[32] I. Antoniadis, *Physics With Large Extra Dimensions*, Beatenberg, 2001, 301; `hep-th/0102202`;
S. Forste, *Fortschr. Phys.* **50** (2002) 221 ;
M. Quiros, `hep-ph/0302189`;
E. Kiritsis,*Fortschr. Phys.* **52** (2004) 200 .

[33] Yu.A. Kubyshin, `hep-ph/0111027`,
J. Hewett, M.Spiropulu, *Ann. Rev. Nucl. Part. Sci.* **52** (2002) 397.

[34] A. Muck, A. Pilaftsis, R. Ruckl, `hep-ph/0209371`.

[35] R. Dick, *Class. Quan. Grav.* **18** (2000) R1 ;
D. Langlois, *Prog. Theor. Phys. Suppl.* **148** (2003) 181;
R. Maartens, *Living Rev. Rel.* **7** (2004) 7 ;
P. Brax, C. van de Bruck, A.C. Davis, *Rept. Prog. Phys.* **67** (2004) 2183 .

[36] C.P. Burgess,*Ann. Phys. (NY)* **313** (2004) 283 .

[37] A. Padilla, `hep-th/0210217`.

[38] G. Gabadadze, `hep-ph/0308112` .

[39] F. Feruglio, *Eur. Phys. J.* **C 33** (2004) S114 .

[40] C. Csaki, hep-ph/0404096;
T.G. Rizzo, `hep-ph/0409309` .

[41] C. Kokorelis, *Nucl. Phys.* **B 677** (2004) 115 [`hep-th/0207234`] .

[42] C.Biggio, F. Feruglio, I. Masina, M. Perez-Victoria, `hep-ph/0305129`.

[43] A. Delgado, A. Pomarol, M. Quiros, *J. High Energy Phys.* **01** (2000) 030 ;
F. del Aguila, J. Santiago, *Phys. Lett.* **B 493** (2000) 175 .

[44] O. DeWolfe, D.Z. Freedman, S.S. Gubser, A. Karch, *Phys. Rev.* **D 62** (2000) 046008 .

[45] M. Gremm,*Phys. Lett.* **B 478** (2000) 434 ; *Phys. Rev.* **D 62** (2000) 044017 .

[46] C. Csaki, J. Erlich, T.J. Hollowood, Y. Shirman, *Nucl. Phys.* **B 581** (2000) 309 .

[47] S. Randjbar-Daemi, M.E. Shaposhnikov, *Phys. Lett.* **B 492** (2000) 361 ;
R. Casadio, A. Gruppuso, *Phys. Rev.* **D 64** (2001) 025020 ;
E.E. Flanagan, S.H.H. Tye, I. Wasserman, *Phys. Lett.* **B 522** (2001) 155 .

[48] A. Kehagias, K. Tamvakis, *Phys. Lett.* **B 504** (2001) 38 ;
S. Kobayashi, K. Koyama, J. Soda, *Phys. Rev.* **D 65** (2002) 064014 .

[49] N.D. Antunes, E.J. Copeland, M. Hindmarsh, A. Lukas, *Phys. Rev.* **D 68** (2003) 066005 ;
I. Oda, *Phys. Lett.* **B 571** (2003) 235 ;
R. Koley and S. Kar, `hep-th/0407158`;
A.V. Yurov, V.A. Yurov, `hep-th/0412036` .

[50] G.R. Dvali, M.A. Shifman, *Phys. Lett.* **B 396** (1997) 64; Erratum-*Phys. Lett.* **B 407** (1997) 452;
G.R. Dvali, G. Gabadadze, M.A. Shifman, *Phys. Lett.* **B 497** (2001) 271.

[51] S.L. Dubovsky, V.A. Rubakov, P.G. Tinyakov, *Phys. Rev.* **D 62** (2000) 105011;
S.L. Dubovsky, V.A. Rubakov, *Int. J. Mod. Phys.* **A 16** (2001) 4331.

[52] T. Gherghetta, M.E. Shaposhnikov, *Phys. Rev. Lett.* **85** (2000) 240;
M. Laine, H.B. Meyer, K. Rummukainen, M. Shaposhnikov, *J. High Energy Phys.* **01** (2003) 068 ; *J. High Energy Phys.* **04** (2004) 027 .

[53] A.A. Andrianov, V.A. Andrianov, P. Giacconi, R. Soldati, *J. High Energy Phys.* **07** (2003) 063 .

[54] R. Rajaraman, *Solitons and Instantons*, North Holland Publ.,1982 .

[55] A. Vilenkin, *Phys. Rept.* **121** (1985) 263.

[56] R. MacKenzie, *Nucl. Phys.* **B B303** (1988) 149.

[57] J.R. Morris, *Phys. Rev.* **D 52** (1995) 1096.

[58] M. Cvetic, H.H. Soleng, *Phys. Rept.* **282** (1997) 159 .

[59] G.E. Volovik, *Sov. Phys. JETP Lett.* **75** (2002) 55 .

[60] D. Bazeia, R.F. Ribeiro, M.M. Santos, *Phys. Rev.* **D 54** (1996) 1852 ;
D. Bazeia, *Braz. J. Phys.* **32** (2002) 869 ;
D. Bazeia, A. S. Inacio, L. Losano, *Int. J. Mod. Phys.* **A 19** (2004) 575 .

[61] A.A. Andrianov, L. Bonora, *Nucl. Phys.* **B 233** (1984) 232, 247 .

[62] T. Shiromizu, K.i. Maeda, M. Sasaki, *Phys. Rev.* **D 62** (2000) 024012 ;
J. Garriga, T. Tanaka, *Phys. Rev. Lett.* **84** (2000) 2778 ;
K.i. Maeda, S. Mizuno, T. Torii, *Phys. Rev.* **D 68** (2003) 024033 .

[63] E.G. Adelberger, B.R. Heckel, A.E. Nelson, *Ann. Rev. Nucl. Part. Sci.* **53** (2003) 77 .

[64] R. Sundrum, *Phys. Rev.* **D 59** (1999) 085009 .

[65] M. Bando, T. Kugo, T. Noguchi, K. Yoshioka, *Phys. Rev. Lett.* **83** (1999) 3601 .

[66] B. Bajc, G. Gabadadze, *Phys. Lett.* **B 474** (2000) 282 ;
C.P. Burgess, R.C. Myers, F. Quevedo, *Phys. Lett.* **B 495** (2000) 384 ;
S. Kachru, M.B. Schulz, E. Silverstein, *Phys. Rev.* **D 62** (2000) 045021 .

[67] S. Forste, Z. Lalak, S. Lavignac, H.P. Nilles, *Phys. Lett.* **B 481** (2000) 360 ;
J. High Energy Phys. **09** (2000) 034 ;
C. Csaki, J. Erlich, C. Grojean, T.J. Hollowood, *Nucl. Phys.* **B 584** (2000) 359 ;
S.P. de Alwis, *Nucl. Phys.* **B 597** (2001) 263 .

[68] O. Corradini, A. Iglesias, Z. Kakushadze, *Int. J. Mod. Phys.* **A 18** (2003) 3221 .

[69] M. Perez-Victoria, *Acta Phys. Polon.* **35** (2004) 2795 .

[70] V.A. Miransky, M. Tanabashi, K. Yamawaki, *Phys. Lett.* **B 221** (1989) 177;
Mod. Phys. Lett. **A 4** (1989) 1043;
W.A. Bardeen, C.T. Hill, M. Lindner, *Phys. Rev.* **D 41** (1990) 1647.

[71] A.B. Kobakhidze, *Phys. Atom. Nucl.* **64** (2001) 941 , `hep-ph/9904203`;
N. Arkani-Hamed, H.C. Cheng, B.A. Dobrescu, L.J. Hall, *Phys. Rev.* **D 62** (2000) 096006.

[72] M. Hashimoto, M. Tanabashi, K. Yamawaki, *Phys. Rev.* **D 64** (2001) 056003;
V. Gusynin, M. Hashimoto, M. Tanabashi, K. Yamawaki, *Phys. Rev.* **D 65** (2002) 116008 ;
Phys. Rev. **D 70** (2004) 096005 .

[73] W. Pauli, *Teoria della Relatività*, Boringhieri, Torino, (1970) pp. 65–66.

[74] L.D. Landau, E.M. Lifshits, *Teoria dei Campi*, Editori Riuniti, Roma (1976) pp. 341–344.

[75] G. Cognola, P. Giacconi, *Phys. Rev.* **D 39** (1989) 2987

[76] D. Vassilevich, *Phys. Rept.* **388** (2003) 279 .

[77] K. Hagiwara *et al.*, *Phys. Rev.* **D 66** (2002) 010001 .

[78] O. Corradini, `hep-th/0405038` .

[79] E.A. Tagirov, `gr-qc/0501026`.

[80] R.M. Cavalcanti, P. Giacconi, R. Soldati, *J. Phys.* **A 36** (2003) 12065 .

[81] W.D. Goldberger, M.B. Wise, *Phys. Rev. Lett.* **83** (1999) 4922; *Phys. Lett.* **B 475** (2000) 275; C. Charmousis, R. Gregory, V.A. Rubakov, *Phys. Rev.* **D 62** (2000) 067505.

[82] S. Bae, P. Ko, H. S. Lee, J. Lee, *Phys. Lett.* **B 487** (2000) 299;
U. Mahanta, A. Datta, *Phys. Lett.* **B 483** (2000) 196 ;
J. Garriga, O. Pujolas, T. Tanaka, *Nucl. Phys.* **B 605** (2001) 192;
C. Csaki, M.L. Graesser, G.D. Kribs, *Phys. Rev.* **D 63** (2001) 065002;
E.E. Boos, Y.A. Kubyshin, M.N. Smolyakov, I.P. Volobuev, `hep-th/0105304`; *Class. Quan. Grav.* **19** (2002) 4591 ;
E.E. Boos, Y.S. Mikhailov, M.N. Smolyakov, I.P. Volobuev, `hep-th/0412204`.

[83] E.W. Kolb, G.Servant, T.M.P. Tait, *J. Cosm. Astropart. Phys.* **07** (2003) 008 .

[84] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, Fifth Edition, Academic Press, (1994), **8.412**2. p. 963.

[85] Robert M. Wald: *General Relativity*, Univ. Chicago Press, (1984).